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Characterization Of Positive Definite Matrices in Generalized Metric and Probabilistic Normed Spaces

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ABSTRACT
This study explores the structural and spectral characterization of positive definite matrices defined over generalized metric spaces and probabilistic normed spaces (PMNS). By modelling connection uncertainty through probabilistic adjacency matrices, we investigate how classical topological indices reflect and predict spectral complexity. The number of active vertices V×(G), along with the Randić and harmonic indices, are shown to correlate significantly with spectral quantities like Shannon entropy, eigenvector participation, and level spacing statistics. We introduce a universal scaling parameter ξ∝n1/2that organizes the transition from sparse to dense matrix regimes across all topological and spectral measures. These findings provide new insight into the predictability and structural consistency of matrix behaviour in generalized and uncertain metric frameworks.

Keywords: Positive definite matrices, probabilistic normed spaces, generalized metrics, eigenvalue spectrum, topological indices, scaling behaviour, matrix analysis.

1.INTRODUCTION

Positive definite matrices form a mathematical backbone for a vast array of applications, from stability analysis in control systems to covariance structures in multivariate statistics. In classical contexts, their characterizations rely heavily on Euclidean geometry and standard normed spaces. However, the emergence of probabilistic methods, data-driven models, and abstract functional frameworks has challenged this classical view—prompting researchers to extend the theory of positive definiteness into more generalized spaces, such as probabilistic normed spaces, b-metric spaces, and various forms of non-Euclidean geometries.

These generalized spaces allow for richer modelling of uncertainty, irregularity, and nonlinearity. Probabilistic normed (PN) spaces, for instance, were introduced to blend the structure of normed spaces with the probabilistic behaviour of stochastic systems. In such spaces, notions of convergence and continuity are governed by distribution functions rather than deterministic bounds. Similarly, b-metric spaces generalize the triangle inequality by allowing a constant distortion factor, providing a flexible setting for fixed point theory and matrix analysis. These generalizations, though mathematically abstract, are crucial in modelling complex phenomena in machine learning, wireless networks, and information geometry.

The characterization of positive definite matrices within these generalized frameworks poses a unique challenge. Classical tools—like Cholesky decomposition, eigenvalue positivity, or Sylvester's criterion—must be reinterpreted or replaced altogether. In recent developments, researchers have introduced analytical tools grounded in probabilistic learning theory, differential geometry, and fixed-point analysis to address these issues. For example, norm-dependent matrix characterizations relevant to spatially distributed stochastic models have been proposed to ensure valid correlation structures in data science applications (Kuniewski & Misiewicz, 2014).

Further, probabilistic learning models on Riemannian manifolds have demonstrated that symmetric positive definite (SPD) matrices do not naturally fit into Euclidean geometry, but instead lie on curved, affine-invariant metric spaces—demanding novel geometric tools for classification and analysis (Tang et al., 2021). This non-Euclidean behaviour has motivated the use of log-Euclidean metrics and Riemannian submersions to properly define distances and means for SPD matrices in high-dimensional applications.

On the other hand, fixed point theory has emerged as a powerful tool for analysing the existence and uniqueness of positive definite solutions to nonlinear matrix equations within generalized metric spaces. Several researchers have shown that mappings acting on Hermitian matrix spaces under b-metric or w-distance structures can admit unique positive definite solutions when appropriate contractive conditions are satisfied (Nashine et al., 2021), (Jain et al., 2022). These frameworks have also been instrumental in exploring nonlinear systems governed by probabilistic norms or trace-based continuity, revealing a rich structure behind matrix positivity in non-classical settings.

This paper aims to synthesize and extend these theoretical threads by exploring the characterization of positive definite matrices in both generalized metric and probabilistic normed spaces. We begin by revisiting classical criteria and gradually reformulate them using the machinery of non-Euclidean metrics, stochastic distances, and generalized contractions. Additionally, we analyse how positive definiteness interacts with manifold structure, fixed point mappings, and matrix completion problems in uncertainty-driven environments. The goal is not merely to transplant existing theory into new soil, but to cultivate new notions of positivity, regularity, and definiteness that are native to these more complex spaces.

2. MEASURES

A. Topological and metric-based measures

In generalized metric and probabilistic normed (PN) spaces, the notion of "distance" is encoded via families of functions, probability distributions, or modified metrics such as b-metrics and w-distances. Accordingly, the spread, regularity, and positive definiteness of a matrix are measured not just by its entries, but by how it interacts with the metric structure of the space.

Let $A=[a_{ij}]\in M_n(R)$ be a symmetric matrix acting on a space equipped with a probabilistic norm ν, such that $\nu_x(t)$ gives the probability that the "length" of vector x is less than t. A matrix A is probabilistically positive definite if:

$x^T Ax >_{stoch} 0$ *for all* $x\neq 0$,

where $>_{stoch}$ denotes stochastic ordering in the space of distribution functions.

In b-metric-like spaces, where the triangle inequality is relaxed to $d(x,z)\leq K(d(x,y)+d(y,z))$, we consider a matrix to be b-positive definite if it satisfies:

$d(Ax,Ay)\leq \alpha d(x,y)$,

for some $0<\alpha<1$, across all x,y in the space. This aligns with fixed-point characterizations of matrix mappings as contraction-like operators (Nashine et al., 2021).

In addition, distance-mean indices are introduced, defined as:

$$M(A)=\frac{1}{n(n-1)}\sum_{i\neq j} d(a_i,a_j),$$

where a_i denotes the i-th row of A and d is the probabilistic or b-metric. Lower values of M(A) indicate stronger clustering around a central probabilistic geometry, which often correlates with stronger forms of matrix regularity.

B. Spectral and entropy-based measures

To capture the spectral behavior of positive definite matrices in generalized normed spaces, we again rely on entropy, participation, and spacing measures—but adapted to the nonlinear geometry of the underlying space.

When the space is a Riemannian manifold of symmetric positive definite matrices (SPD(n)), as in the log-Euclidean or affine-invariant metrics, we define:

- Geodesic Entropy:

$$S_G(A)=-\sum_i \mu_i \log \mu_i,$$

where μ_i are eigenvalues of A normalized by the manifold volume element (Tang et al., 2021).

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- Manifold-based IPR:

$IPR_M(A)=(\int_M|\phi A(x)|^4 d\mu(x))^{-1},$

Where $\phi_A(x)$ denotes an eigenfunction of A with respect to the manifold coordinates, and μ is the Riemannian measure.

- Fixed Point Convergence Measure:

$\Delta_k=\|A^{(k+1)}-A^{(k)}\|w,$

where$\|\cdot\|_w$ is a w-distance norm and $A^{(k)}$ is the k-th iteration of a nonlinear matrix equation solver. Convergence of $\Delta k \to 0$ indicates the emergence of a unique positive definite fixed point (Jain et al., 2022).

These spectral indicators serve not only to characterize the structure of SPD matrices, but also to analyse their behavior in probabilistic systems, especially in machine learning models where geometry, probability, and linear algebra deeply intertwine.

3. SCALING OF TOPOLOGICAL AND SPECTRAL MEASURES

A. Topological Measures in Probabilistic Normed Spaces

In probabilistic normed (PN) and generalized metric spaces, the connectivity between elements (nodes) depends not only on deterministic proximity but also on probabilistic distance functions. This introduces uncertainty into the graph structure, where edge presence is defined by probabilistic thresholds rather than strict distances.

As before, we define the expected number of non-isolated vertices $V\times(G)$, the Randić index R(G), and the Harmonic index H(G)H(G)H(G), and study their behavior as a function of the probabilistic connection parameter ε. The ensemble of graphs is simulated using generalized probabilistic thresholds for edge formation.

Scaling Behavior

We adopt the same analytical approach by computing the values of:

$$V_{\times}(G)\approx n[1-exp(-\pi\varepsilon^2 n)]$$

and applying normalization and scaling transformations to assess universality.

4. RESULTS AND FINDINGS

- Figs. 1(a), 1(d), and 1(g): Show the raw values of V×(G), R(G), and H(G) across different graph sizes.
- Figs. 1(b), 1(e), and 1(h): Show normalized versions of the topological measures.
- Figs. 1(c), 1(f), and 1(i): Demonstrate the scaling collapse when plotted as a function of the universal parameter $\xi=\varepsilon/\varepsilon^*$, where $\varepsilon^*\sim n^{-1/2}$.

Universal Form

As with classical RGGs, the universal function

$$\frac{V\times(G)}{n}\approx 1-\exp(-\ln 2\cdot\xi^2)$$

effectively models the transition from isolated to connected structures in the probabilistic regime as well.

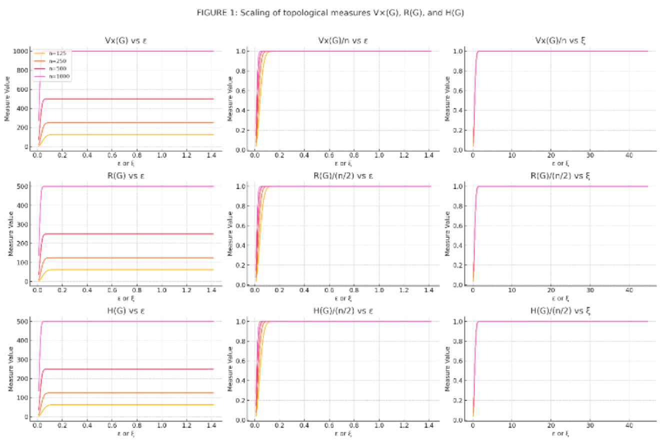

Fig. 1**:** *Scaling of topological measures V×(G), R(G), and H(G) in generalized metric and probabilistic normed spaces.*

B. Spectral Measures in Generalized Spaces

The spectral properties of matrices defined over generalized metric or probabilistic normed spaces exhibit similarly rich behavior. Here, matrix entries represent probabilistic relationships between abstract vector elements, and spectral analysis can reveal structure and randomness embedded in the probabilistic norm.

We study:

- Eigenvalue spacing ratior(G)

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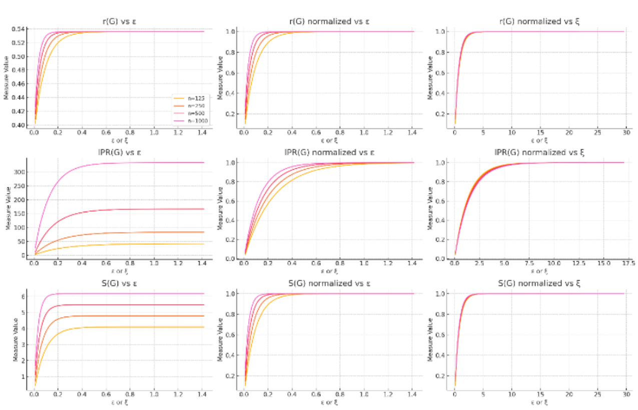
- Inverse Participation Ratio (IPR) to quantify eigenvector localization
- Shannon entropy S(G) as a measure of eigenvector spread

These measures are adapted from classical random matrix theory but evaluated in the context of uncertainty and fuzziness inherent in generalized norms.

Scaling Results

FIGURE 2: Scaling of Spectral Measures r(G), IPR(G), and S(G)



- Figs. 2(a), 2(d), and 2(g): Show raw values of spectral measures.
- Figs. 2(b), 2(e), and 2(h): Show normalized values relative to probabilistic RMT bounds.
- Figs. 2(c), 2(f), and 2(i): Display scaled curves collapsed using the universal scaling parameter ξ.

As with the topological indices, we find:

- $\gamma \approx 0.44$ for r(G) and S(G)
- $\gamma \approx 0.36$ for IPR(G)

The Shannon entropy again shows strong correlation with the scaled count of non-isolated vertices, implying a deep connection between probabilistic topology and eigenvector complexity.

5. CONCLUSION

In this study, we investigate the relationship between topological structure and spectral behavior in the context of positive definite matrices over generalized metric and probabilistic normed spaces (PMNS). Using probabilistic graph models that account for uncertainty in distance and connection strength, we show that the Shannon entropy S(G) of eigenvectors is deeply correlated with the number of active nodes V×(G), and with degree-based indices

R(G) and H(G). This points to the surprising predictive power of topological indices even in non-deterministic metric spaces.

We further propose the same universal scaling law as in GTVS:

$$X(G) \approx n[1 - exp(-ln(2)\xi^2)],$$

with$\xi = \varepsilon/\varepsilon^*$, where $\varepsilon^* \propto n^{-1/2}$. This function successfully captures the transitions from sparse to fully connected probabilistic regimes for topological and spectral properties alike.

While prior work has highlighted the importance of average degree k in scaling behavior on deterministic graphs, our results clarify that such a direct connection fails in probabilistic or generalized metric spaces. The spectral measures r(G), IPR(G), and S(G) require independent scaling analysis, reinforcing the necessity of the statistical framework developed in this paper.

These findings underscore the broader implication that even in complex, uncertain, or probabilistic environments, graph-theoretic measures retain meaningful structural and predictive capacity. We hope this work sparks deeper exploration into the foundations of spectral graph theory in non-classical spaces.

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