Stability Analysis of Boundary Layer Fluid Flow Under Variable Pressure Gradients

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ABSTRACT

This study is a mathematical exploration of the stability properties of two-dimensional laminar flow in the boundary layers under the conditions of variable pressure gradient. Flow of an incompressible Newtonian fluid over a flat plate is taken into consideration and the governing boundary layer equations are developed under the usual considerations of the boundary layers. With a power-law change of the external velocity, similarity transformations are used to simplify the governing partial differential equations to the Falkner Skan type ordinary differential equation of the simplest flow profile. The linear stability theory is then used through superimposition of small perturbation onto the fundamental flow and the equations governing the flow are then linearized giving an eigenvalue problem similar to the Orr Sommerfeld formulation. The stability behavior is examined in the form of the parameter of pressure gradient that controls the nature of the external flow. The findings indicate that positive pressure gradients yield rounded velocity fields and inhibit the growth of disturbances which, in turn, increases the stability of flows but adverse pressure gradients promote the growth of disturbances and makes it more likely that the separation of the boundary layer will take place. The discussion is of the importance of pressure gradients in the control of stability of the boundary layer and offers theoretical knowledge applicable in aerodynamic design, control of flow dynamics, as well as prediction of transition effects in viscous flow.

Keywords: Boundary Layer Flow, Stability Analysis, Pressure Gradient, Similarity Solution, Laminar Flow

Introduction

Boundary layer theory is one of the foundational areas of fluid mechanics and applied mathematics, and a strong method of analysis of viscous fluid flow around solid boundaries [1]. Boundary layer analysis has made a major contribution to the simplification of the NavierStokes equations since Prandtl, gave the seminal work that has made it possible to isolate regions where viscous effects are dominant over inertial forces [2]. The strategy has played a critical role in elucidating a broad spectrum of flow phenomena found in aerodynamics, hydrodynamics, heat and mass transfer and many other industrial and technological applications [3].

In the real flow scenarios, fluid flow along solid surfaces is not often exposed to a uniform set of the free-stream conditions [4]. The difference in geometry, curvature of the surface, external forces or acceleration of the flow always leads to pressure gradients in the direction of flow. These pressure disparities have a potent effect on the behaviour and design of the boundary layer. Favorable pressure gradient (greater external velocity over the surface) is more likely to thin the boundary layer and inhibit the growth of disturbances [5]. Conversely, a negative pressure gradient, which occurs due to the slowdown of flow, facilitates the increase of the boundary layer, creates distortion on velocity profile and can result in flow segregation and instability. Consequently, the research of the stability of the boundary layers in the variable pressure gradients is of general significance both in the theoretical and practical fluid mechanics [6].

Mathematically, the stability of the boundary layer flows is also inherently associated with time developments of small perturbation overlaying a fundamental laminar flow. Linear stability theory offers a systematic framework to be used to study whether these perturbations decay or increase with time [7]. The critical information on the occurrence of instability and changes in laminar to more complicated flow regimes are obtained as a result of the eigenvalue problems. The analyses are critical in the prediction of flow separation, an increase in aerodynamic drag, and a loss of efficiency in engineering systems [8].

Similarity solutions take a central stage in the theory of the boundary layer since solutions using similarity can reduce nonlinear partial differential equations of a viscous flow to ordinary differential equations. In the case of Newtonian fluids, the classical Blasius solution is a simplest solution that is associated with a zero pressure gradient flow over a flat plate [9]. Boundary layer flows under the influence of pressure gradients caused by the variations of the external velocity of a power law nature are described by the more general FalknerSkan similarity solutions. These solutions not only give an insight into the velocity distribution in the boundary layer, but also form the basis base flows such that stability analysis can be carried out.

It is familiar that similarity solutions can only be found to a restricted class of exterior velocity distributions [10]. The presence and character of these solutions not only hinge crucially on the parameter of pressure gradient encompassing the effect of the external flow field. The changes in this parameter cause major changes in the velocity profile of the boundary layer, with inflection even appearing in them which have close relations with flow instability. This, therefore, makes the interpretation of the boundary layer stability as influenced by the parameter of pressure gradient a relevant mathematical issue.

The stability properties of variable pressure gradient, boundary layer flows, has become the focus of continuous research since the starting years because of their applicability in prediction of transition and separation processes. Stabilizing forces in general, unfavorable pressure differences stabilize the flow by decreasing shear along the wall, whereas favorable pressure differences amplify shear and increase disturbances. These competing effects render the behavior of stability very sensitive to the flow conditions changes and require a comprehensive mathematical inquiry.

The current work is related to the stability examination of steady, two-dimensional laminar boundary layer flow of incompressible Newtonian fluid under varying conditions of the pressure gradient. Similarity variables are found based on a power-law external velocity field, which converts the governing equations of a boundary layer into the governing similarity equation [11]. The resulting equations of the Falkner-Skan type are those of the basic flow profile. The temporal dynamics of small perturbations about this fundamental flow are then studied using the linear perturbation theory, and then provides an eigenvalue formulation which defines the flow stability.

This work is aimed at coming up with a clear mathematical explanation on the stabilizing and destabilizing influences of favorable and adverse pressure gradients to the flow of the boundary layers. The discussion indicates how parameters of pressure gradient play out in regulating the growth of disturbances and stability of flows. The obtained results can be applied to a very broad field of applications such as in the optimization of aerodynamic surfaces, boundary layer control methods and in the prediction of viscous flow separation and transition.

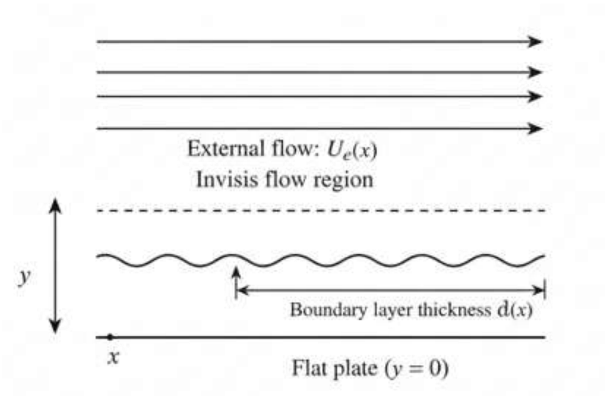

Figure 1: Schematic diagram of boundary layer flow under variable pressure gradient

Figure 1 is an idealized two-dimensional laminar boundary layer growing over a flat plate with the external velocity Ue(x) changing in the streamwise direction, thus creating a pressure gradient and stability in the boundary layers is affected.

Governing Equations

The current analysis assumes a two-dimensional laminar, steady and incompressible flow of a Newtonian fluid and laminar in the boundary layer over a smooth flat plate at a constant external pressure gradient [12]. The Cartesian coordinate system is selected in such a way that the plate is aligned with the x-axis in the direction of the flow and, the y-axis is along the normal direction of the plate surface. The velocity components in the x- and the y-direction respectively are denoted by u(x,y) and v(x,y).

In the case of incompressible flow, the continuity equation is the conservation of mass and therefore the velocity field must satisfy the conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Which makes the fluid density unchanged all over the flow field.

Momentum conservation equations are deduced using the Navier-Stokes equations. The flow under assumptions of the boundary layer is defined by the small viscous layer that is located near the surface where the change in velocity in the normal direction is significantly higher than the change in velocity in the stream-wise direction [13]. Also, the pressure in the boundary layer is taken to be equal to the pressure exerted in the outer inviscid flow. Consequently, the streamwise momentum equation will become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{dP_e}{dx} \qquad (2)$$

with ν being the kinematic viscosity of the liquid, ρ being the constant density and $P_e(x)$ being the external pressure distribution along the plate.

The pressure gradient term which exists in equation (2) is due to the inviscid flow outside of the boundary layer and is determined by the equation of Bernoulli. Bernoulli law gives the relation in case of steady inviscid flow

$$\frac{1}{\rho}\frac{dP_e}{dx} = -U_e(x)\frac{dU_e(x)}{dx} \qquad (3)$$

where $U_e(x)$ is the outward velocity at the edge of the layer of boundary. This relationship is the determinant of the fact that the pressure gradient across a boundary layer is completely determined by the spatial change in the external flow velocity [14].

Replacing equation (3) in equation (2) the boundary layer momentum equation can be written in explicit form as a function of the external velocity field as;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + U_e(x)\frac{dU_e(x)}{dx} \qquad (4)$$

This type of the equation shows clearly the impact of acceleration or deceleration of the external flow on the internal dynamics of the boundary layer directly on the pressure gradient term.

Boundary conditions used in the current problem are prescribed on physical consideration. The fluid clinging to solid boundary at the surface of the flat plate is because of the viscosity, and no fluid will penetrate through the surface. The no-slip and impermeability conditions are, therefore, of the form

$$u = 0, v = 0 \ at \ y = 0 \qquad (5)$$

The greater the distance of the plate, the more viscous effects become less significant and the velocity in the layer of the boundary should approaches to the same velocity of the external inviscid flow as much as possible. Thus far-field boundary condition can be expressed as

$$u \to Ue(x) as y \to \infty \qquad (6)$$

The mathematical model of the steady two-dimensional two-boundary layer flow under a variable pressure gradient can be described as a whole using equations (1), equation (4) accompanied by the boundary conditions (5) and (6). This equation is the preferred expression of the necessary balance between inertia, viscosity and pressure forces in the boundary layer and is the initial basis of the similarity transformations and linear stability analysis that have been established in the ensuing sections [15].

Similarity Transformation

The equations of governing obtained in the section above are coupled nonlinear partial equations of the independent variables x and y. Such equations are usually hard to treat analytically. But given some flow conditions the equations can be simplified to ordinary differential equations by using similarity transformations. The fundamental point of similarity analysis is that the spatial dependence of the velocity field can be modeled in a single independent variable, assuming that the external flow has a defined functional form [16].

In order to carry out this strategy, a stream function ψ(x,y) is added wherein:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}, \qquad (7)$$

The above definition makes sure that the continuity equation is identically satisfied on all admissible selections of psi(x,y), and thus only a smaller set of governing equations has to be thought.

The external inviscid flow is supposed to depend on the plate in a power-law distribution,

$$U_e(x) = U_0 x^m, \qquad (8)$$

where $U_0>0$ is a positive constant and mmm is a real parameter [17]. This expression of the external velocity is generic enough to describe a large set of physically significant pressure gradient situations. Specifically, the m>0 cases are accelerating flows due to positive pressure gradients, m=0 cases are the uniform free-stream velocity due to the classical flat-plate boundary layer, and cases m<0 are decelerating flows due to negative pressure gradients.

Due to the dimensional reasons and the need to compress the velocity profile into one curve, the similarity variable η and the stream function ψ are defined as

$$\eta = y\sqrt{\frac{U_e(x)}{\nu x}}, \qquad \psi = \sqrt{\nu x U_e(x)} f(\eta), \qquad (9)$$

with ν the kinematic viscosity of the fluid and f(η) a similarity function in a dimensionless form, to be ascertained.

Replacing the definition of the ψ, in equation (9), into equation (7), the components of the velocity may be written explicitly as functions of the similarity function. With respect to y, differentiation is

$$u = U_e(x) f'(\eta), \tag{10}$$

that the streamwise velocity is directly proportional to the external velocity. In a similar manner differentiation with respect to x will yield the normal velocity component as [18].

$$v = \frac{1}{2}\sqrt{\frac{\nu U_e(x)}{x}}\left[(m+1)\eta f'(\eta) - f(\eta)\right], \tag{11}$$

where primes denote differentiation with respect to η.

Replacement of the expressions (10) and (11) of the velocities in the boundary layer momentum equation alongside the pressure velocity relation derived by Bernoulli equation, leads to a cancellation of all explicit dependence on the streamwise coordinate, x. The resulting governing partial differential equation is simplified to a third-order nonlinear ordinary differential equation which is either of Falkner-Skan type, below

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{12}$$

The dimensionless parameter

$$\beta = \frac{2m}{m+1} \tag{13}$$

is a parameter that is referred to as the pressure gradient parameter, which represents the effect of the variation in the external velocity on the flow in the boundary layer. The value of β and its sign determine the shape of the profile of velocities, and have a significant impact on stability characteristics of the flow.

The physical boundary conditions put on the field of velocities are reduced to the following conditions on the similarity function:

$$f(0) = 0, f'(0) = 0, f'(\infty) = 1 \tag{14}$$

These conditions are respectively the no-slip condition at the wall, the normal flow not at the surface and the condition that the boundary layer velocity be a smooth matching of the external inviscid flow far away at the plate [19].

The equations (12)-(14) give a nonlinear boundary value problem whereby the solution would be the base velocity profiles in case of variable pressure gradients in the boundary layer flow. These solutions require critical value of the parameter of pressure gradient 8 as a condition of existence and quality. In more detail, adverse pressure gradients sufficiently large can result in the loss of physically significant solutions indicating the separation of the boundary layers.

In order to get a better interpretation of the association between the power-law index mmm and the pressure gradient parameter β, illustrative values are summed up in Figure 2.

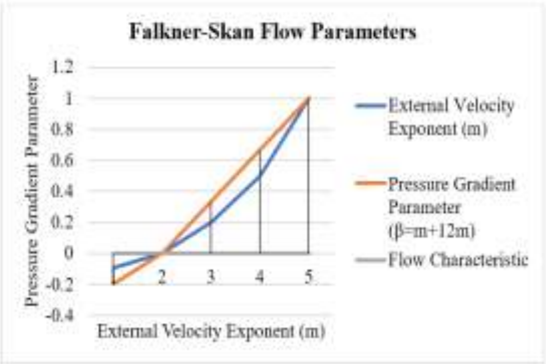

Figure 2: Relation between external velocity exponent mmm and pressure gradient parameter β

The similarity transformation thus offers a strict mathematical construction of the minimization of the original equations of the boundary layer to a single ordinary differential equation which represents the key effects of the pressure gradients [20]. The similarity solutions obtained are the fundamental flow profiles in the stability analysis that is done in the following section.

Stability Analysis

Stability of the laminar boundary layer flow is studied analyzing the behavior of small disturbances applied to the basic steady flow. The basic supposition of the linear stability theory is that the behavior of the system to the perturbations of an adequate small magnitude can be linearized by the governing equations. $u_0(y)$ and $v_0(y)$ are the streamwise and normal velocity factors of the basic flow expressed by the similarity solution of the Falkner-Skan equation [21].

This overall velocity field may be written as a combination of the basic flow and a minor deviation in the form:

$$u(x,y,t) = u_0(y) + \epsilon u_1(x,y,t), \tag{15}$$

$$v(x,y,t) = v_0(y) + \epsilon v_1(x,y,t), \tag{16}$$

where ϵ is a small, dimensionless parameter satisfying $0<\epsilon\ll 1$, and $u_1(x,y,t)$, $v_1(x,y,t)$ represent the perturbation components. The assumption of small ϵ is that the interaction between perturbations nonlinearly is negligible in the first degree.

Using equations (15) and (16) as substitutes into the governing equations in the boundary layer then discarding all terms of order $O(\epsilon^2)$ and above, we obtain the linearized disturbance equations. The perturbation field linearized continuity equation is as follows:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{17}$$

The linearized streamwise momentum equation is obtained to be as follows:

$$\frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + u_1 \frac{\partial u_0}{dy} = v \frac{\partial^2 u_1}{\partial y^2} \tag{18}$$

The terms in equation (18) are clearly physically interpreted (the first term is related to unsteady effects, second and third terms is the advection of disturbance by the basic flow, fourth term is related to the interaction of the disturbance with the base flow shear, and the last term is the viscous diffusion) [22].

In order to explore the time behaviour of disturbances, normal mode solutions are assumed in classical form.

$$u_1(x,y,t) = \hat{u}(y) e^{i(\alpha x - \omega t)}, \tag{19}$$

$$v_1(x,y,t) = \hat{v}(y) e^{i(\alpha x - \omega t)}, \tag{20}$$

where α is the real streamwise wave number, ω=ωr+iωi is the complex frequency, and uˆ(y), vˆ(y) are the amplitude functions of the perturbations. The real part ωr represents the phase speed of the disturbance, while the imaginary part ωi determines its growth or decay rate.

Substitution of equations (19) and (20) into the linearized equations (17) and (18) reduces the problem to an eigenvalue formulation for ω. Using the continuity equation to eliminate vˆ(y), the governing equation for the disturbance amplitude uˆ(y) can be written symbolically as

$$L(\hat{u}) = \omega \hat{u}, \tag{21}$$

where L a linear differential operator in which $u_0(y)$ is the basic velocity profile, ν is the kinematic viscosity, and α is the wave number. This operator equation is structurally similar to the classical Orr-Sommerfeld equation, which is a key operator in the hydrodynamic stability theory [23].

It is the sign of the imaginary value of the eigenvalue ω which is used to ascertain the linear stability of the flow. If

$$\omega_i < 0, \tag{22}$$

the perturbations die away exponentially as time and flow go and the flow is said to be linearly stable. Conversely, if

$$\omega_i > 0, \tag{23}$$

the perturbations become exponentially growing and this represents the linear instability of the boundary layer flow [24].

The influence of the pressure gradient on the stability is included by the simplest velocity profile $u_0(y)$, which is explicitly dependent on the parameter of the pressure gradient β. Positive pressure gradients (β>0) produce more complete velocity profiles with less shear near the wall, thus inhibiting the growth of disturbances. On the contrary, negative pressure gradients (β<0) cause a reduction in velocity and can cause inflection points in the velocity profile. Through classical inflection-point theory, such profiles can be more prone to instability and can lead to separation of the boundary layers. The general effect the parameter of the pressure gradient has on the stability of the boundary layer is resumed in Table 1 showing the passage of the stable to the highly unstable regimes as the parameter of the pressure gradient, 0, diminishes.

Table 1: Influence of Pressure Gradient Parameter (β) on Boundary Layer Stability

Range of Pressure Gradient Parameter (β)	Flow Regime / Acceleration Type	Stability Level	Physical Interpretation
β < 0	Strong deceleration (inflectional velocity profile)	Highly Unstable	Adverse pressure gradient leads to flow separation and strong amplification of disturbances
β ≈ 0	Transition point	Marginally Stable	Neutral stability; corresponds to Blasius-type flow
0 < β < 0.4	Mild acceleration (Fuller velocity profile)	Stable	Favorable pressure gradient suppresses disturbance growth
β ≥ 0.4	Strong acceleration (Highly stable profile)	Strongly Stable	Strong favorable pressure gradient significantly stabilizes the boundary layer

Besides the effect of the pressure gradient effect, the wave number of the disturbance also greatly affects the stability of the boundary layer. Table 2 shows a high amplification in unstable regimes of disturbances with moderate wave numbers, and the death of disturbances with very small and/or very large wave numbers, respectively, when subjected to insufficient shear interaction and viscous damping, respectively. This qualitative interdependence of the growth of disturbances between their wave number is a typical property of the stability of a boundary layer.

Table 2: Qualitative dependence of disturbance growth rate ω_i on wave number α

Wave number α	Stability response
Small α	Weak amplification or decay
Moderate α	Maximum amplification (if unstable)
Large α	Strong viscous damping

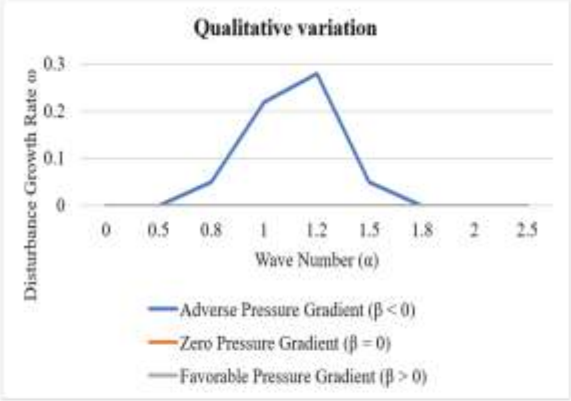

Figure 3: Qualitative variation of disturbance growth rate

The combined effects of pressure gradient and wave number on disturbance growth are illustrated in Figure 3, which shows the qualitative variation of the disturbance growth rate ω_i with wave number α for adverse pressure gradients ($\beta<0$), the growth rate becomes positive over a finite range of α, indicating instability. For zero pressure gradient flow ($\beta=0$), the growth rate remains close to the neutral stability line $\omega_i=0$, corresponding to marginal stability. In contrast, for favorable pressure gradients ($\beta>0$), ω_i remains negative for all values of α, demonstrating complete suppression of disturbance growth and enhanced flow stability [25].

Results and Discussion

The stability calculation that has been conducted in the current study makes it clear that the decisive role of the pressure gradients on the laminar boundary layer flow behavior is clear. Through the use of similarity transformations and linear perturbation theory, the governing equations were simplified to a mathematically solvable form that enables straightforward analysis of disturbance growth in a variety of conditions dictated by pressure gradient. The obtained results give not only mathematical clarity but also physical understanding of the stabilizing and destabilizing forces that are inherent in the flows along the boundary layers.

The stabilizing effect of favorable pressure gradients is one of the important results of the analysis. In case the parameter of pressure gradient β is positive, the external flow accelerates along the surface and this causes fuller velocity profiles in the boundary layer. Mathematically these profiles have lower velocity gradients in the area close to the wall and this reduces the amplification of disturbances by shear. The imaginary component of the disturbance eigenvalue is therefore negative at all wave numbers, which is a sign of decay of perturbations and a linear flow is stable. This is in line with the classical stability theory and it proves the fact that positive pressure gradients will suppress the development of instability.

Conversely, when the pressure gradient is adverse ($\beta<0$) the structure of the basic flow is considerably changed. The reduction in the external velocity results in the augmentation of the boundary layer and augmented shear at the wall. Mathematically, it is seen in velocity profiles which can have inflection points, an aspect that has been seen to be tightly linked with instability by inflection-point theory. The stability analysis shows that the growth rate of disturbances will become positive in a finite range of wave numbers, which means that the perturbation will amplify and eventually destroy the laminar flow. This finding gives a straightforward theoretical justification to the behavior of adverse pressure gradients to facilitate boundary layer separation.

The fact that stability is also dependent on the wave number also increases the interpretation of the results. It was observed that disturbances with moderately high wave numbers were best amplified in unstable regimes and very small and very large wave numbers are suppressed because of inadequate interaction of shear and very high viscous damping respectively. This qualitative behavior, as presented in Table 2 and depicted in Table 1, is also in agreement with the known findings in the field of hydrodynamic stability theory and supports the soundness of the given formulation.

The next key point in the discussion is why the pressure gradient parameter β as a unifying control parameter. Geometry and a variance of the external velocity together act on the boundary layer to produce a combined effect, β, which is captured by the Falkner Skan similarity solutions. The discussion reveals that slight alterations in the β can result in the large alterations in stability of the flows, which demonstrates how sensitive the behavior of the boundary layer is to external factors. This sensitivity highlights the role of taking care of the model of pressure differences in both theoretical and computational analyses of boundary layer flows.

Conclusion

The present research has involved a detailed mathematical examination of the stability of the laminar flow of two-dimensional flow of the boundary layer in condition of variable pressure gradient conditions. The governing equations of the boundary layer were simplified by using similarity transformations, and the basic flow structure could be investigated systematically in the Falkner-Skan formulation. Linarear stabiliy theory was then used to study the time-dependent behaviour of small perturbations to the base flow and an eigenvalue problem that defines stability in the flow was produced. It has been found out that positive pressure gradients stabilize the boundary layer through generation of fuller velocity profiles and inhibition of growth of perturbations, whereas negative pressure gradients contribute to instability and definition of boundary layer separation. These are in congruence with the classical boundary layer theory and the significance of the pressure gradients in controlling the behavior of the flow is confirmed. The findings of the research are relevant to the further theoretical comprehension of stability in a boundary layer and present the useful information to design and control of engineering systems where the flow driven by pressure gradients is met.

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