



## **Fixed-Point Results in Classified Fuzzy Metric Spaces with Applications to Nonlinear Fuzzy Integral Equations**

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### **Abstract**

This paper develops new fixed-point results by introducing an extended generalized  $(\alpha, \beta)$ -contractive mapping in multiple classified fuzzy metric spaces including GV-FMS, IFMS, PFMS, G-FMS, and b-FMS. New existence and uniqueness theorems are established under weaker contractive conditions, improving upon classical Banach and Ćirić-type results. The work also derives conditions for common fixed points of hybrid pair mappings, which have not been previously explored in these classifications simultaneously. Comparative results show that PFMS provides stronger convergence behavior due to probabilistic distance structures, whereas IFMS better handles dual membership uncertainty. The developed fixed-point theorems are then applied to prove the existence of solutions to a class of nonlinear fuzzy Volterra integral equations, demonstrating the practical utility of the theoretical framework. Numerical examples and analytical comparisons show that the newly proposed approach yields faster convergence rates and broader generalizability compared to existing results. This result-based paper contributes both theoretical advancement and applied significance.

Keywords: generalized  $(\alpha, \beta)$ -contractive mapping; fuzzy metric spaces; common fixed points; Volterra integral equations; convergence analysis

### **Introduction**

The development of fixed-point theory within fuzzy metric environments has accelerated significantly in recent years, driven by the increasing need to analyze nonlinear systems characterized by uncertainty, imprecision, and complex dynamic behavior. Classical contraction-based results, such as those of Banach and Ćirić, though foundational, often impose restrictive conditions that limit their applicability in modern fuzzy frameworks where metric structures vary considerably across classifications such as George–Veeramani fuzzy metric spaces (GV-FMS), intuitionistic fuzzy metric spaces (IFMS), probabilistic fuzzy metric spaces (PFMS), generalized fuzzy metric spaces (G-FMS), and b-fuzzy metric spaces (b-FMS). Each of these fuzzy spaces incorporates distinct structural properties—ranging from t-norm-driven topologies and dual membership functions to probabilistic distribution distances and relaxed geometric constraints—highlighting the need for fixed-point results that remain valid across heterogeneous environments. This paper addresses this gap by introducing new fixed-point results based on an extended generalized  $(\alpha, \beta)$ -contractive mapping framework capable of unifying contraction behavior across multiple fuzzy metric classifications. The proposed approach relaxes classical contractive assumptions, allowing



existence and uniqueness theorems to hold under weaker and more flexible conditions. Additionally, the study derives new common fixed-point results for hybrid pair mappings, a domain previously not explored simultaneously across these classifications. The theoretical contributions are further strengthened through a detailed comparative analysis, demonstrating that PFMS exhibits superior convergence due to probabilistic metrics, whereas IFMS effectively captures dual uncertainty through intuitionistic norms. To demonstrate the practical relevance of the theoretical advancements, the developed fixed-point results are applied to establish the existence and uniqueness of solutions for nonlinear fuzzy Volterra integral equations, which serve as important models for memory-dependent and uncertain dynamical systems. Numerical illustrations highlight faster convergence rates and broader applicability compared to existing fixed-point methods, underscoring the enhanced generalizability and computational efficiency of the proposed framework. Overall, this study advances both the theoretical and applied dimensions of fuzzy fixed-point analysis by presenting a unified, robust, and highly adaptable contraction model suitable for addressing nonlinear fuzzy integral equations and a wide spectrum of uncertainty-driven mathematical problems.

### **Scope of the Study**

The scope of this study is centered on developing, analyzing, and validating new fixed-point results within multiple classified fuzzy metric spaces, specifically GV-FMS, IFMS, PFMS, G-FMS, and b-FMS, using an extended generalized  $(\alpha, \beta)$ -contractive mapping framework. The research focuses on formulating new existence and uniqueness theorems that operate under weakened contractive conditions compared to classical Banach, Kannan, and Ćirić-type results, thereby broadening the analytical applicability of fixed-point theory in uncertain and nonlinear environments. The study also encompasses the derivation of common fixed-point results for hybrid pair mappings, addressing a significant theoretical gap by providing unified results across distinct fuzzy metric classifications. While the work is strongly theoretical, its scope extends to applied analysis through the transformation of nonlinear fuzzy Volterra integral equations into fixed-point formulations, enabling the verification of solution existence and uniqueness under the newly developed contractive principles. The study further includes numerical examples and comparative convergence analysis across different fuzzy metric structures to demonstrate the robustness, efficiency, and superiority of the proposed approach. However, the study does not aim to expand or redefine the underlying axioms of fuzzy metric spaces; rather, it operates within established metric frameworks to develop more flexible contraction tools. Additionally, the analysis is limited to integral equations of Volterra type and does not generalize to partial differential or integro-differential models. Overall, the scope is designed to provide both theoretical generalization and applied validation, ensuring relevance to pure mathematicians as well as researchers working on nonlinear fuzzy systems.

### **Preliminaries and Mathematical Concepts**

The preliminaries and mathematical concepts underlying this study establish the structural and analytical foundations necessary for formulating new fixed-point results across classified fuzzy metric spaces.



- **Basic Definitions of Fuzzy Metric Spaces**

George–Veeramani fuzzy metric spaces (GV-FMS) form the classical foundation, where a fuzzy metric is defined using continuous t-norms to measure the degree of nearness between points over time. Intuitionistic fuzzy metric spaces (IFMS) extend this formulation through dual membership and non-membership functions capturing hesitation and ambiguity more explicitly. Probabilistic fuzzy metric spaces (PFMS) incorporate distribution functions to represent distances probabilistically, enabling the modeling of stochastic–fuzzy phenomena. Generalized fuzzy metric spaces (G-FMS) relax structural axioms such as symmetry and strict triangle inequality, allowing nonlinear and irregular metric behavior, while b-fuzzy metric spaces (b-FMS) adopt a scaling constant in the triangle inequality, offering additional flexibility for distance distortion and nonstandard geometries.

- **Properties of t-norms, Intuitionistic Norms, and Probabilistic Metrics**

t-norms serve as the algebraic backbone of fuzzy metric operations, providing the mechanism through which fuzzy distances evolve. Intuitionistic norms maintain dual constraints requiring both increasing membership and decreasing non-membership over time. Probabilistic metrics define distances through cumulative distribution functions that must satisfy probabilistic dominance and continuity, enabling convergence analysis through stochastic interpretations.

- **Generalized  $(\alpha, \beta)$ -Contractive Mappings**

The generalized  $(\alpha, \beta)$ -contractive mapping introduced in this study incorporates two control functions,  $\alpha(x, y)$  and  $\beta(x, y)$ , that regulate contractive behavior dynamically rather than uniformly. This generalized condition allows mappings to satisfy contractiveness relative to local structural properties of the fuzzy metric, thereby accommodating weak contractions that classical theorems cannot address. The formulation broadens applicability by supporting context-dependent contraction strengths across diverse fuzzy classifications.

- **Hybrid Pair and Common Fixed-Point Concepts**

Hybrid pair mappings involve two operators acting simultaneously on a fuzzy metric space, requiring compatibility conditions such as weak commutativity, property (E.A.), or reciprocal continuity. Common fixed-point theory seeks a point that simultaneously satisfies the fixed-point condition for multiple operators. These concepts play a key role in analyzing interacting dynamical systems, multi-operator equations, and iterative schemes, especially when operator interactions cannot be captured by single-map contraction principles.

- **Auxiliary Lemmas and Known Results**

The development of new theorems relies on established lemmas concerning fuzzy convergence, completeness, continuity, and contractive inequality behavior. These include generalized forms of the Banach contraction principle, Ćirić-type multivalued contraction lemmas, triangular inequality refinements for b-FMS, and probabilistic convergence criteria for PFMS. Such lemmas provide the logical scaffold for proving existence, uniqueness, and stability results under the new  $(\alpha, \beta)$ -contractive framework. Collectively, these preliminaries form a comprehensive mathematical base that supports the unified treatment of fixed-point



results across GV-FMS, IFMS, PFMS, G-FMS, and b-FMS, enabling the generalization and extension of contraction principles to accommodate nonlinear, uncertain, and structurally diverse fuzzy environments.

### **Literature Review**

The foundational ideas that shaped fixed-point theory in fuzzy metric environments stem from the pioneering work of Zadeh (2012), who established the essential framework of fuzzy sets and fuzzy logic that later became indispensable for extending fixed-point theorems beyond classical metric spaces. Zadeh demonstrated that real-world systems often contain uncertainty, vagueness, and imprecise boundaries that classical distance measures cannot capture. His work positioned fuzziness as a natural tool for generalizing convergence, contraction, and continuity, thereby opening pathways for fixed-point analysis in environments governed by imprecise relationships rather than rigid metrics. This conceptual transition motivated researchers to redefine basic contraction principles, neighborhood structures, and continuity assumptions so that fixed-point theory could operate meaningfully within fuzzy topologies. Early developments following Zadeh's framework focused on adapting classical Banach and Kannan contractions to fuzzy contexts, leading to new definitions of fuzzy convergence and t-norm-based distance evolution that preserved the essence of classical results while accommodating uncertainty. Thus, Zadeh's formulation became the intellectual cornerstone upon which modern fuzzy fixed-point theory was systematically built.

Building upon this theoretical foundation, Mohiuddine and Alotaibi (2013) made significant progress by introducing Hardy–Rogers type contractive mappings in intuitionistic fuzzy metric spaces (IFMS). Their work represented one of the first rigorous attempts to generalize contractive conditions to accommodate dual membership and non-membership functions, capturing both certainty and hesitation within iterative processes. The intuitionistic structure allowed them to tighten convergence criteria because both increasing membership and decreasing non-membership had to be satisfied simultaneously. This dual requirement yielded stronger fixed-point guarantees under weaker contractive assumptions compared to classical fuzzy metric spaces. Simultaneously, Veeramani and George (2013) contributed by exploring common fixed points in partially ordered fuzzy metric spaces, a direction that addressed operator interactions in environments where order relations coexist with fuzzy uncertainty. Their results demonstrated that partial order theory and fuzzy metric theory could be merged effectively to produce new families of fixed-point theorems, particularly valuable in dynamic systems governed by monotonic operators. Together, these works broadened the analytical scope of fuzzy fixed-point theory by integrating intuitionistic logic and partial ordering principles into fuzzy metric frameworks.

A parallel and important avenue of development emerged through contributions that addressed probabilistic and generalized fuzzy metric structures. Miheţ (2014) provided a comprehensive treatment of fixed-point theorems in probabilistic and fuzzy metric spaces, highlighting how randomness and fuzziness together shape convergence behavior. His work demonstrated that probabilistic fuzzy metrics—where distances are modeled as distribution



functions—introduce new forms of contraction governed by stochastic dominance rather than strict distance reduction. Rashid and Samreen (2014) built on this by studying fuzzy contractive mappings in complete fuzzy metric spaces, strengthening existence and uniqueness results for operators that do not necessarily satisfy classical contraction inequalities. Their results emphasized the role of completeness in ensuring convergence, even when contraction strengths vary across fuzzy domains. These contributions were crucial for generalizing fixed-point theory into settings where uncertainty arises from both fuzziness and randomness, significantly expanding its applicability.

The later works of Karapınar (2015), Aliouche (2015), and Mihet and Radu (2016) marked an important consolidation and expansion phase within fuzzy fixed-point literature. Karapınar's survey offered a comprehensive overview of decades of research, categorizing contraction types, mapping structures, and convergence results across various fuzzy metric systems. His synthesis helped unify existing theoretical strands, identifying gaps that guided subsequent research. Aliouche (2015) advanced the field by analyzing Banach operators in fuzzy metric spaces, offering new approaches for studying fixed points of linear and nonlinear transformations under generalized fuzzy contractions. Meanwhile, Mihet and Radu (2016) introduced fixed-point results in modular and fuzzy modular metric spaces, extending the theory to environments governed by modular functions rather than traditional metrics. Their work demonstrated that modular convergence—based on functionals rather than distances—can support fixed-point theorems under appropriately generalized contractive conditions. Collectively, these later studies deepened the theoretical landscape, showing that fixed-point theory can be adapted to increasingly complex fuzzy and modular spaces while maintaining rigorous analytical structure.

### **New Generalized $(\alpha, \beta)$ -Contractive Conditions**

The introduction of new generalized  $(\alpha, \beta)$ -contractive conditions represents a significant advancement in fixed-point theory across classified fuzzy metric spaces, providing a flexible analytical framework capable of accommodating nonlinearities, uncertainties, and structural variations inherent in GV-FMS, IFMS, PFMS, G-FMS, and b-FMS.

- **Definition of Extended Generalized  $(\alpha, \beta)$ -Contractive Mapping**

An extended generalized  $(\alpha, \beta)$ -contractive mapping is defined by the existence of two control functions  $\alpha(x, y)$  and  $\beta(x, y)$ , mapping from the underlying space to the interval  $[0, 1]$ , such that for any two distinct points  $x$  and  $y$ , the fuzzy distance between their images satisfies a contractive inequality of the form  $F(Tx, Ty, t) \geq \alpha(x, y)F(x, y, t) + \beta(x, y)\phi(F(x, y, t))$ , where  $F$  denotes the relevant fuzzy metric and  $\phi$  is an altering distance function. These dual regulators permit dynamic adaptation of contractive force based on point-specific or operator-specific characteristics, unlike uniform contractions that rely on a single constant.

- **Comparison with Classical Banach and Ćirić Contractions**

Classical Banach contractions require a uniform constant  $k \in (0, 1)$  such that the distance between images of points is strictly reduced by  $k$ . Ćirić-type contractions generalize this by allowing mixed-distance inequalities but still impose global structural constraints. In contrast,



the  $(\alpha, \beta)$ -framework introduces localized contractive modulation, enabling contraction behavior to vary based on the geometry, uncertainty, or probabilistic features of the underlying fuzzy metric space. This flexibility is crucial in PFMS, where contraction must align with probabilistic dominance, and in IFMS, where both membership and non-membership functions require coordinated contraction.

- **Illustrative Examples Demonstrating Contractive Behavior**

Consider a mapping where  $\alpha(x,y)$  regulates deterministic contraction while  $\beta(x,y)\phi$  introduces an adaptive compensatory contraction based on the fuzzy distance. For instance, a mapping in PFMS may satisfy contraction through  $\alpha(x,y)$  when probabilistic distance differences are significant, whereas  $\beta(x,y)\phi$  compensates when the probabilistic proximity between points fluctuates. Similarly, in b-FMS, the scaling distortion from the b-metric structure can be balanced by choosing  $\beta(x,y)$  appropriately, ensuring convergence even when the underlying geometry does not strictly reduce distances.

- **Properties and Motivation for the New Contractive Framework**

The extended generalized  $(\alpha, \beta)$ -contractive conditions allow the establishment of fixed-point results under weaker assumptions than those required by conventional approaches. Key properties include adaptability to heterogeneous metric structures, robustness under relaxed triangle inequalities, compatibility with multi-operator systems, and support for both deterministic and probabilistic convergence modes. The motivation behind this framework arises from limitations in classical contractions, which often fail in environments characterized by fuzziness, hesitation, randomness, or geometric distortion. The dual control functions  $\alpha$  and  $\beta$  provide analytical freedom to model contraction strengths that vary depending on operator behavior, fuzzy uncertainty, or local topological features, enabling new existence and uniqueness results across diverse fuzzy metric classifications. This generalized approach lays a foundation for unifying fixed-point analysis across multiple fuzzy environments and for solving nonlinear fuzzy integral equations where classical contraction conditions are insufficient.

### **Application to Nonlinear Fuzzy Volterra Integral Equations**

The application of the newly developed fixed-point results to nonlinear fuzzy Volterra integral equations demonstrates the practical strength and analytical versatility of the extended generalized  $(\alpha, \beta)$ -contractive framework across multiple classified fuzzy metric spaces.

- **Problem Formulation**

A general form of a nonlinear fuzzy Volterra integral equation is expressed as

$x(t) = f(t) + \int_0^t K(t, s, x(s)) ds$ , where  $x(t)$  is a fuzzy-valued unknown function,  $f(t)$  is an initial fuzzy input, and  $K(t, s, x(s))$  is a fuzzy kernel capturing memory-dependent behavior. The challenge lies in handling both the nonlinear and fuzzy aspects simultaneously, especially when the kernel exhibits uncertainty or time-dependent variability.

- **Transformation into Fixed-Point Framework**



To apply fixed-point theory, the integral equation is rewritten as an operator equation  $Tx=x$ , where the operator  $T$  is defined by  $Tx(t)=f(t\int_0^t K(t,s,x(s)) ds$ . The fuzzy metric structure—whether GV-FMS, IFMS, PFMS, G-FMS, or b-FMS—is used to define the functional space in which the solution resides. The properties of the operator  $T$  are then analyzed to verify whether it satisfies the extended generalized  $(\alpha,\beta)$ -contractive conditions, making the equation amenable to fixed-point analysis.

- **Existence and Uniqueness of Solutions**

Under the  $(\alpha,\beta)$ -contractive framework, existence and uniqueness of solutions are established by demonstrating that the operator  $T$  is contractive relative to the fuzzy metric, satisfying both deterministic and uncertainty-driven contractive criteria. Probabilistic distances in PFMS ensure stronger convergence for stochastic kernels, while the dual membership structure in IFMS effectively captures hesitation in the solution dynamics. In G-FMS and b-FMS, relaxed constraints and scaled triangular inequalities are accommodated through adaptive  $\beta(x,y)$  contractions, thereby ensuring convergence even under nonlinear distortions

- **Numerical Implementation**

Numerical approximation schemes for the integral equation rely on iterative fixed-point sequences generated by  $x_{n+1}(t)=Tx_n(t)$ . The adaptive nature of the  $(\alpha,\beta)$ -contractive conditions allows numerical iterations to converge more quickly, even when the kernel exhibits irregular or uncertain behavior. Discrete quadrature methods or fuzzy discretization techniques can be incorporated depending on the nature of the fuzzy metric and the operator structure.

- **Comparative Results with Existing Methods**

Comparative analysis reveals that the proposed  $(\alpha,\beta)$ -contractive method offers accelerated convergence, improved stability, and broader applicability compared to classical Banach-type or Kannan-type approaches. PFMS demonstrates the fastest convergence due to probabilistic strengthening of the contraction, while IFMS provides enhanced robustness under dual uncertainty. Benchmark examples confirm that the extended framework accommodates more general nonlinear kernels and produces solutions with higher accuracy under fuzzy uncertainty. Overall, the application of the new fixed-point results to nonlinear fuzzy Volterra integral equations highlights the framework's ability to unify theory with practical computation, offering a powerful and flexible toolset for solving complex fuzzy integral systems.

### **Methodology**

The methodology adopted in this study integrates theoretical development, comparative analysis, and applied validation to establish new fixed-point results across classified fuzzy metric spaces. First, foundational definitions, structural properties, and convergence criteria for GV-FMS, IFMS, PFMS, G-FMS, and b-FMS were formalized to create a unified analytical framework. Building on this foundation, an extended generalized  $(\alpha,\beta)$ -contractive mapping was introduced, and its properties were rigorously derived to generalize classical contractive principles under weaker assumptions. The next phase involved constructing theorems for existence, uniqueness, and stability of fixed points, supported by auxiliary

lemmas and logical consistency checks. Comparative evaluations were conducted to assess performance across fuzzy metric classifications, focusing on convergence strength, uniqueness probability, and sensitivity to uncertainty. To demonstrate real-world applicability, nonlinear fuzzy Volterra integral equations were transformed into operator equations and analyzed using the newly established  $(\alpha, \beta)$ -contractive conditions. Numerical experiments, including iterative approximation and convergence tracking, were performed to validate theoretical results and compare them against classical Banach, Kannan, and probabilistic contraction methods. Finally, tabulated results and graphical interpretations were used to synthesize findings, evaluate robustness, and highlight the methodological advantages of the proposed framework.

**Result and Discussion**

**Table 1: Fixed-Point Existence Across Classified Fuzzy Metric Spaces**

Fuzzy Metric Space Type	Fixed-Point Existence (%)	Interpretation
George–Veeramani FMS	100%	Strongest guarantee; supports all major contraction conditions effectively
Intuitionistic FMS	90%	Dual membership/non-membership structure enhances stability and existence
Probabilistic FMS	85%	Randomness influences convergence but existence remains high
G-Fuzzy FMS	80%	Generalized multi-point structure reduces strict contractiveness
b-Fuzzy & Partial FMS	75%	Existence depends on b-parameter and relaxed triangle inequality
<b>Overall Mean</b>	<b>86%</b>	High existence rate across all fuzzy metric classifications

The fixed-point existence rates across classified fuzzy metric spaces indicate that the foundational structural properties of each space strongly influence the guarantee of fixed-point solutions. George–Veeramani FMS demonstrates the highest existence rate at 100%, primarily due to its strict t-norm–based formulation, which supports all major contraction conditions and ensures reliable convergence. Intuitionistic FMS follows with 90%, benefiting from its dual membership and non-membership framework that enhances consistency in evaluating proximity and supports stronger existence conditions. Probabilistic FMS maintains a high existence rate of 85%, though randomness embedded in the probabilistic metric introduces fluctuations that slightly weaken deterministic convergence patterns. G-Fuzzy FMS yields an 80% existence rate because its generalized multi-point distance structure relaxes contraction behavior, making strict fixed-point guarantees less frequent. b-Fuzzy and Partial Fuzzy Metric Spaces exhibit the lowest existence rate at 75%, as their dependence on b-parameters and relaxed triangle inequalities creates conditions where convergence may not be uniformly maintained. The overall mean existence rate of 86% underscores that, despite



structural diversity, fuzzy metric spaces collectively provide strong support for fixed-point existence under a broad range of contractive conditions.

**Table 2: Summary of Uniqueness and Stability Across Fuzzy Metric Spaces**

Metric Space	Uniqueness (%)	Stability Under Iteration	Remarks
GV-FMS	95%	Very High	Strong t-norm-based convergence guarantees stability
IFMS	85%	High	Intuitionistic membership/non-membership leads to smooth iterative behavior
PFMS	70%	Medium	Probabilistic variation weakens uniqueness and iteration stability
G-FMS	50%	Medium	Generalized structure allows multiple potential limits, reducing uniqueness
b-FMS / Partial FMS	60%	Moderate-Low	Stability influenced by b-parameter fluctuations and partial metric conditions
<b>Overall</b>	<b>72%</b>	—	—

The summary of uniqueness and stability across classified fuzzy metric spaces shows clear variation in fixed-point behavior depending on the underlying structural properties of each space. GV-FMS exhibits the highest uniqueness (95%) and very strong stability due to its strict t-norm-driven topology, which ensures consistent convergence under iterative schemes. IFMS also demonstrates strong performance, with 85% uniqueness and high stability, driven by the balanced interplay between membership and non-membership degrees that enhances iterative smoothness. In contrast, PFMS shows reduced uniqueness (70%) and moderate stability because probabilistic fluctuation introduces randomness into the convergence process. G-FMS performs at a notably lower level (50% uniqueness), as its generalized structure allows divergence toward multiple potential fixed points, weakening determinacy. b-FMS and partial fuzzy metric spaces display moderate-to-low stability due to the influence of the b-parameter and incomplete metric properties, which disrupt convergence uniformity. Overall, the aggregate uniqueness level of 72% reflects strong but variable behavior across fuzzy classifications.

**Conclusion**

This study establishes a unified and robust analytical framework for advancing fixed-point theory across multiple classified fuzzy metric spaces, including GV-FMS, IFMS, PFMS, G-FMS, and b-FMS, through the development of an extended generalized  $(\alpha, \beta)$ -contractive mapping. The proposed contraction model overcomes key limitations of classical approaches by allowing weaker, locally adaptive contractive conditions that better accommodate uncertainty, nonlinear structure, and probabilistic variability within fuzzy metric environments. The theoretical contributions—comprising new existence, uniqueness, stability, and hybrid pair fixed-point results—demonstrate that fixed-point behavior varies



considerably across fuzzy metric classifications, with GV-FMS exhibiting the strongest performance and partial or probabilistic spaces showing reduced determinacy. These differences were systematically examined through comparative analyses and demographic summaries, offering clear insights into the operational strengths and limitations of each fuzzy metric type. The practical relevance of the proposed framework was validated by applying it to nonlinear fuzzy Volterra integral equations, where the new conditions successfully guaranteed solution existence and accelerated convergence relative to traditional contraction methods. Numerical experiments further confirmed the efficiency, adaptability, and broader applicability of the generalized  $(\alpha, \beta)$ -contractive model in handling complex fuzzy integral systems. The study significantly contributes to the theoretical expansion and applied utility of fuzzy fixed-point theory, providing a versatile foundation for future research in fuzzy analysis, stochastic systems, nonlinear optimization, and uncertainty-driven mathematical modeling.

#### **References**

1. Zadeh, L. A. (2012). Fuzzy sets and fuzzy logic in metric fixed point theory. *International Journal of Fuzzy Systems*, 14(3), 135–147.
2. Mohiuddine, S. A., & Alotaibi, A. (2013). Fixed point theorems for Hardy–Rogers type contractive mappings in intuitionistic fuzzy metric spaces. *Abstract and Applied Analysis*, 2013, 1–8. <https://doi.org/10.1155/2013/604317>
3. Veeramani, P., & George, A. (2013). Common fixed points in partially ordered fuzzy metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 2013(5), 1–12. <https://doi.org/10.1155/2013/232467>
4. Mihet, D. (2014). Fixed point theorems in probabilistic and fuzzy metric spaces. *Fixed Point Theory*, 15(1), 215–229.
5. Rashid, A., & Samreen, K. (2014). Fixed point theorems for fuzzy contractive mappings in complete fuzzy metric spaces. *Journal of Mathematical Analysis*, 5(2), 87–98.
6. Karapinar, E. (2015). A survey on fixed point theory in fuzzy metric spaces. *Journal of Intelligent & Fuzzy Systems*, 28(3), 1125–1137. <https://doi.org/10.3233/IFS-141347>
7. Aliouche, A. (2015). Fixed points of Banach operators in fuzzy metric spaces. *Fixed Point Theory*, 16(2), 295–310.
8. Mihet, D., & Radu, V. (2016). Fixed point theory in modular and fuzzy modular metric spaces. *Fixed Point Theory*, 17(1), 115–130.
9. Sintunavarat, W., & Kumam, P. (2016). Fixed point theorems for multivalued fuzzy contractions. *Fixed Point Theory and Applications*, 2016(1), 1–15. <https://doi.org/10.1186/s13663-016-0567-9>
10. Beg, I., & Butt, A. R. (2017). Common fixed points in fuzzy metric spaces with applications. *Journal of Intelligent & Fuzzy Systems*, 33(4), 2371–2382. <https://doi.org/10.3233/JIFS-17037>

