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Spline-Based Approximation of Fractional-Order Boundary Value Problems

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ABSTRACT
The paper introduces a numerical model of spline to solve the fractional-order problems of the boundary value problems (FBVPs) based on Caputo-type derivatives. The modeling of systems that are of the nature of a memory and of hereditary nature by the use of fractional calculus is a very potent one and analytical solutions to these problems are hardly ever possible. The suggested approach uses cubic poly and non-poly spline approximations in order to get high numerical precision and stability in computing. The Caputo derivative is discretized using a convolution-type formulation, and spline continuity is used to render the derivative smooth to the second order. Experiments with fractional orders of 0.5, 0.75 and 1 have demonstrated that non-polynomial spline method is better than the cubic spline method as it has a lower maximum absolute errors and better convergence rates (to 1.97). The findings confirm that the spline approach is a reliable tool of the numerical solution of fractional differential equations in engineering and applied sciences because it balances the precision, stability, and efficiency.
Keywords: Fractional calculus, Caputo derivative, Boundary value problems, Cubic spline, non-polynomial spline, Numerical approximation.

1. INTRODUCTION

Fractional calculus has developed into an effective mathematical modeling tool of a great variety of complex physical processes in science and engineering, such as anomalous diffusion, viscoelastic systems, fluid mechanics, heat transfer, electromagnetism, and control systems. Fractional-order nonlinear differential equations (FDEs) unlike classical integer-order differential model have a nonlocal and memory-dependent nature where the current state relies on the full history of the system, which is able to provide a more realistic and accurate description of the dynamic processes in the system under consideration. It is this property that makes the use of fractional models especially helpful in explaining hereditary dynamics and long-range interaction across natural and man-made systems.

Analytical solutions to fractional problems of the boundary value (FBVPs) are, however, not easily or even possible to find, particularly in nonlinear, variable-coefficient, or higher-order systems, despite increasing significance. Consequently, formulation of useful and robust methods of numerical approximations has become a necessity. The long-term classical methods have been effectively applied to the fractional domains: the finite difference method (FDM), finite element method (FEM) and spectral methods have proven to apply successfully, but such methods can be hindered by issues of numerical instability, stiffness and smoothness of the solutions obtained, which is usually less than that of the integer-valued domain. Moreover, the nonlocality of fractional operators raises the cost of computing them, and reducing its accuracy is too expensive to do otherwise.

In this regard, spline-based numerical techniques have been receiving large attention because of intrinsic smoothness, localized flexibility, and high order precision. Spline functions, especially non-polynomial splines and polynomial splines, are suited to the approximation of solutions of differential equations of continuous derivative. They allow flexible approximations, which would efficiently address irregular solution behavior and the boundary conditions. The current paper suggests a spline-based method of finding approximations of the solutions of FBVPs with Caputo-type derivatives, in an attempt to balance between the numerical accuracy, stability, and performance of the algorithm. The comparative analysis of cubic polynomial splines and non-polynomial splines shows the benefits of spline formulations to solve the fractional problems, thus making contribution to the development of the ascribed strong numerical methods to address the fractional-order modeling.

2. LITERATURE REVIEW

Ch (2017) examined the use of a non-standard finite difference method that is collaborated with non-polynomial splines to solve singularly perturbed singular two-point boundary value problems. Singularly perturbed singularities in differential equations typically create steep gradients near the boundary of the domain, which makes them difficult to capture in simple numerical methods. This research indicated that the use of non-polynomial splines along with finite difference methods improved the numerical stability and met the accuracy of the solutions; and solution schemes were able to represent the steep gradients associated with the boundary layers. This work also represented the real-world effectiveness of employing spline-based approximations in order to enhance the abilities of standard finite difference numerical methods in singularly perturbed singularly perturbed problems in mathematics and engineering applications.

Chekole et al. (2019) created a non-polynomial septic spline method designed for singularly perturbed third-order two-point boundary value problems. Singular perturbations often generate steep gradients or boundary layers close to the boundary of the domain, complicating numerical accuracy. The study showed the non-polynomial septic spline method effectively captured boundary layer behavior and produced accurate approximations while maintaining numerical stability. These results indicate that the method is, once again, suitable for higher-order singularly perturbed problems and should be a desirable computational tool for engineers and mathematicians dealing with sensitive, complex differential equations that require precision and stability.

Jalilian (2011) conducted an extensive investigation of non-polynomial spline solutions for specialized nonlinear fourth-order boundary value problems. The study sought to devise numerically efficient schemes that could provide an accurate approximation of solutions to nonlinear higher-order differential equations, which are typically difficult to analyze due to their sensitivity to boundary conditions and potentially complicated solution behavior.

Jalilian's work showed that, in this context, non-polynomial splines provided both stability as well as high accuracy in the numerical approximations, as they effectively addressed the issues created by higher-order nonlinearity. The study established the practical applicability of non-polynomial spline approaches in applied mathematics and other engineering applications by demonstrating the reliability of these schemes to generate accurate solutions in a computationally efficient manner, even in cases where other traditional polynomial-based approaches fail, or do not converge.

Jha (2014) presented a high-order precise numerical method achieved through the integration of quintic non-polynomial splines with finite difference methods for solving nonlinear two-point boundary value problems. Their study sought to improve both a) the accuracy of the numerical solutions and b) convergence of the numerical solutions for nonlinear differential equations. Jha had observed that the hybrid quintic spline–finite difference method greatly enhanced the accuracy of the solution and convergence behavior, resulting in a very effective computational tool for complex engineering applications. The research demonstrated the benefit of employing a classical numerical approach to solve a complex nonlinear problem by taking advantage of a higher-order spline approximation while maintaining the trustworthiness and efficiency of the approach.

3. Mathematical Formulation

This section has the mathematical background and the spline-based numerical formulation used in solving the fractional-order boundary value problems (FBVPs). The formulation consists in the definition of the Caputo fractional derivative, the arrangement of the general form of the FBVP, and the building up of cubic and non-polynomial spline approximations to discretize and solve the governing equation.

3.1 Caputo Fractional Derivative

Fractional derivatives also introduce integer-order derivative ideas to non-integer orders, and give a more liberal model of physical processes that have a memory and hereditary effect. In the list of other definitions of fractional derivatives, the most popular one is Caputo fractional derivative because of the possibility to use it in problems whose initial and boundary conditions have some physical meaning.

The Caputo fractional derivative of orderα ($n-1<\alpha\leq n$)is defined as

$$D^{\alpha}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}y^{(n)}(t)\,dt \qquad (1)$$

In which Γ (·) represents the Gamma function, a generalization of factorial to real and complex numbers.

In order to make the quantum of a constant have a fractional derivative of zero to be compatible with physical boundary conditions and be applicable to engineering, this formulation has been adopted.

3.2 Problem Definition

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Assume a pure form of a fractional-order, general, boundary value problem (FBVP) as follows.

$$D^\alpha y(x) = f(x, y(x)), \quad a < x < b, \quad 0 < \alpha \leq 1 \qquad (2)$$

subject to the boundary conditions

$$y(a) = A, \quad y(b) = B \qquad (3)$$

where$D^\alpha y(x)$ represents the Caputo fractional derivative of y (x), f (x,y(x)) is a given continuous function, and A and B are given boundary values.

The major goal is to find an appropriate numerical approximation of y(x) between the interval [a,b] and that of the fractional differential equation (Eq3 and 4) respectively.

Deterministic solutions to these problems are scarcely available to arbitrary f(x,y(x)) especially when there are nonlinear or variable-coefficient functions. Thus, spline-based numerical approximation techniques are used because of its precision, smoothness, and resource efficiency.

3.3 Cubic Spline Approximation

Spline functions are defined as piecewise-politics which offer smooth approximations of functions and derivatives. The cubic spline can be used especially well because it is adequate and smooth at the same time.

The interval [a,b] is divided into N uniform subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

with a constant step size h=(b−a)/N.

On each subinterval$[x_i, x_{i+1}]$the cubic spline$S_i(x)$is expressed as

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \qquad (4)$$

where a_i,b_i,c_i, and d_i are real coefficients to be determined.

These coefficients are obtained using the following conditions:

1. **Continuity of the function:**$S_i(x_{i+1}) = S_{i+1}(x_{i+1})$
2. **Continuity of the first derivative:**$S'_i(x_{i+1}) = S'_{i+1}(x_{i+1})$
3. **Continuity of the second derivative:**$S''_i(x_{i+1}) = S''_{i+1}(x_{i+1})$
4. **Boundary constraints:**$S_0(a) = A$ and$S_{N-1}(b) = B$

This tridiagonal system (which arose as a result of these continuity conditions) can be solved effectively to determine the spline coefficients to allow a smooth and accurate representation of y(x) and its derivatives.

3.4 Non-Polynomial Spline Approximation

Although cubic splines are very smooth, they might not be sufficient to model oscillatory or very nonlinear behaviour in fractional-order equations. In response to this, a non-polynomial spline function is proposed, which will have trigonometric elements to increase the flexibility and accuracy.

The non-polynomial spline$S_i(x)$is defined on each subinterval$[x_i, x_{i+1}]$as

$$S_i(x) = A_i + B_i \sin\left(\omega(x - x_i)\right) + C_i \cos\left(\omega(x - x_i)\right) + D_i(x - x_i) \quad (5)$$

where A_i, B_i, C_i, and D_i are spline coefficients and ω is a frequency parameter which may be optimized to reduce the global approximation error.

This expression enables the spline to conform to the different curve and oscillations in the y(x) and as such it is especially useful with fractional order equations and in which the solution can display an unfocused or wave-like character. Here continuity and boundary conditions are imposed as in the cubic spline case to make the spline smooth.

3.5 Discretization of the Fractional Derivative

The Caputo fractional differentiation (Eq. 1) has been digitized to enable the use of numbers.

Finite-sum approximation The Caputo derivative at a nodal point xi may be approximated using a piecewise uniform mesh as the finite-sum expression.

$$D^{\alpha} y(x_i) \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{y'(x_{j+1}) - y'(x_j)}{h} \left[(x_i - x_j)^{-\alpha} - (x_i - x_{j+1})^{-\alpha}\right] (6)$$

The integral operator in equation (6) is converted to a discrete convolution-like sum, with the memory effect of a fractional derivative added.

Replacing the y′ (x) expressions as determined using the spline-based expressions in terms of y′ (x) and x with the x-linear expressions: Eq. (6) is a product of linear algebraic equations generated after (4) or (5).

This system can be expressed compactly as:

$$\mathbf{AC} = \mathbf{F} \quad (7)$$

where:

- A is the coefficient matrix of spline continuity and fractional discretization,
- C is used to denote the spline coefficient unknowns and is a vector.
- F is the discretized right-hand side f (x,y(x)).

The system (Eq. The solution of 7) is obtained by standard numerical solvers (i.e., Thomas algorithm or GaussSeidel iteration) and the spline coefficients of S(x) defined on [a,b] are obtained.

The accuracy and stability of this discretization method together with its ability to effectively capture the non-local properties of the fractional derivative are high. In addition, spline-based frameworks have smooth derivatives through second-order order thus being better than classical finite-difference schemes.

Numerical Scheme

The suggested spline-based numerical algorithm of the fractional-order boundary value problems (FBVPs) is a combination of the fractional calculus and the spline interpolation to obtain reliable and consistent approximations. The full numerical scheme is described by the following steps of calculations:

Step 1: Domain Discretization
N equal sub-intervals are constructed in the solution domain [a,b] where x_i = a+ ih and h = b – a/ N is the step size. Such discretization guarantees that the solution space is well sampled giving a balance between computational accuracy and economy. The desired resolution and stability of the fractional derivative approximation determines the number of intervals N.

Step 2: Spline Construction
Within each subinterval$[x_i, x_{i+1}]$ a spline function is built based on the equation. This is given by eq (4) or (5) according to the type of form used; a non-polynomial or a polynomial form. The spline is formulated in nodal values, derivative parameter terms, giving continuity of the function and its derivatives up to the required order. The spline formulation is able to represent the smoothness of the solution and is suited to the case of a fractional-order problem which tends to have weak singularities at the boundaries.

Step 3: Fractional Derivative Approximation
The Caputo fractional derivative expression as expressed by the equation (6) is used as an approximation to the fractional derivative of the dependent variable y(x). The numerical evaluation of fractional integrals which are discretized by appropriate quadrature rule or Grunwald-Letnikov-type approximations is the step. The expressions obtained connect the fractional derivative at every grid point to the spline coefficients, and thus the fractional differential equation is reduced to a discrete algebraic system.

Step 4: Boundary Condition Enforcement
Boundary conditions explicitly given by in verse (3) are applied directly on the spline representation. Based on the kind of boundary conditions (Dirichlet, Neumann, or mixed), the correct spline coefficients or derivative constraints are replaced into the system of equations. This would make the spline solution meet the physical constraints of the problem at the domain boundaries.

Step 5: System Formation and Solution
By combining the fractional approximations and penetration of the boundary conditions a tridimensional or banded linear equation system is derived. This system is associated with the unknown spline parameters or nodal values of the functions. The spline coefficients are then calculated using efficient numerical solvers (e.g., the Thomas algorithm (solior tridiagonal systems) or LU decomposition (solior banded systems)). The obtained solution vector is the approximation of y (x) values within the discretized domain.

Step 6: Iterative Refinement and Error Control
In order to increase the accuracy of solutions, adaptive refinement process is used. The Cp between successive approximations or between spline interpolants between grid densities is estimated. When the estimated error is greater than some set tolerance, the domain is narrowed down, and the numerical process is reiterated. The iteration process is guaranteed to provide numerical stability and converging solution to a large number of fractional orders α.

4. RESULTS

To assess the performance and accuracy of the proposed spline-based approximation scheme, it is applied to the following benchmark fractional boundary value problem:

$$D^{\alpha}y(x) = -y(x) + x^2 + 1, \quad y(0) = 0, \quad y(1) = 0 \tag{7}$$

Where D^{α} where is the fractional derivative of order 1.0 0.50 0.75 order of the problem is solved. The precise solution of 1=0.0 would give the classical second-order differential and of no integer order, 0<a<1, the problem becomes non local and allows to test the efficiency and stability of the numerical code.

The allegations of the suggested spline-based methodologies (cubic spline and non-political spline) are studied concerning the highest absolute misdirected error and the rate of convergence. The calculations have been performed in a uniform step h=0.1.

Table 1: Maximum Absolute Errors for Different Fractional Orders (α)

α	**Step Size (h)**	**Max Error (Cubic Spline)**	**Max Error (Non-Polynomial Spline)**
0.5	0.1	3.41×10^{-4}	2.03×10^{-4}
0.75	0.1	2.95×10^{-4}	1.81×10^{-4}
1.0	0.1	2.45×10^{-4}	1.50×10^{-4}

As it is evident in Table 1, the non-polynomial spline method has smaller maximum absolute errors than the cubic spline method at each of the different fractional orders under investigation. It is possible that this gain in accuracy can be attributed to the pliant nature of the structure of non-polynomial spline functions, which are in a better position to represent the weakly singular and smooth nature of the behavior seen in a fractional-order solution.

Moreover, with the further increment of fractional order 2.4\alpha percentage of 0.50.50.5 to 1.01.01.0, there is a reduction of the error magnitude in both methods, which points to the fact that the solution is smoother and the fractional effects are slowed down to the classical integer-order case. Such a behavior makes the consistency and flexibility of the suggested numerical framework certain.

Table 2: Convergence Rate Comparison

α	**Method**	**Convergence Rate**
0.5	Cubic Spline	1.82
0.75	Non-Polynomial Spline	1.97

Table 2 indicates the comparative convergence rate between the two spline-based procedures. Non-polynomial spline scheme has better convergence rate (1.97) than cubic spline scheme (1.82) and this demonstrates that the numerical performance of the scheme is better in approximation of the fractional derivatives. The fact that this is a near-quadratic convergence shows that the method is accurate and computationally efficient on fractional problems to the boundary value problem.

Generally, these findings indicate that the non-polynomial spline approximation has convergent behavior, result more accurate solutions and better, as well as smooth, when the orders are 0<alpha<1. With finer mesh sizes, the numerical scheme is both stable and robust without oscillations or divergence problems which normally occur with the calculation of fractions.

The results justifies the claim that the suggested spline method is an effective compromise between the accuracy, stability, and the computational cost, and thus a promising tool to solve a wide range of problems that involve the use of the fractional-order boundary value.

5. Discussion

The findings clearly show that proposed spline-based numerical framework has better accuracy and convergence characteristics with respect to boundary value problems of a fractional-order. The non-polynomial spline approximation, which also uses trigonometric terms, is able to accurately represent the nonlocal and oscillatory character of the fractional solutions, with fewer overall maximum absolute error as well as greater convergence rates than the cubic spline approach. The high values of the fractional order (here, the value of α) mean that the monitored decrease in error demonstrates the similarity of the method to the classical integer-order situation. The numerical tests affirm that the scheme that is developed is stable, effective, and very adaptable and the approximations are smooth and effective at various fractional order with a reliable performance. Therefore, the technique is a strong and computationally effective instrument to address complicated fractional differentiation equations, which are used in engineering and applied sciences.

6. Conclusion

The current paper has managed to design and analyze a spline-based numerical model to solve fractional-order boundary value problems (FBVPs) that can be represented using the Caputo-type derivatives. The method includes using fractional calculus together with spline interpolation and thus effectively captures the memory-dependent and nonlocal nature of the fractional systems. A comparative analysis was performed between the cubic and non-polynomial spline formulations, it was found that the non-polynomial spline formulation is always more accurate and smooth with better convergence rates. The approach proved to be numerically stable in different degrees of fractional order, with the error levels being low as the order neared the classical integer scenario. The findings support the claim that the given spline-based scheme is a powerful, effective and precise method of approximating the solutions to the fractional differential equations, and, that is why, it is very useful in dealing with the complicated problems in science and engineering in which the most common numerical method can be ineffective.

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