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Solving Linear Diophantine Equations Using Generalized Chinese Remainder Theorem

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ABSTRACT
This project presents a new framework for solving linear Diophantine equations by using Generalized Chinese Remainder Theorem (GCRT). Standard approaches are often computationally heavy and require deep understanding of advanced number theory. To address this, we build on Ming Xiong's "minimum principles"- a function based technique for simplification – to develop an algorithm that converts a system of linear Diophantine equations into a GCRT problem.

Crucially, our method handles these systems regardless of whether the moduli are coprime. By combining Xiong's concepts of minimum-principle-based transformations with the GCRT framework, our algorithm simplifies the solution process, making it both more accessible and computationally efficient. Ultimately this study provides a more generalized and practical tool for finding integer solutions, addressing a significant gap in the existing literature. The proposed method demonstrates how fundamental mathematical principles can be applied in new contexts to simplify complex problems, offering a clear and straightforward alternatives to conventional methods.

KEYWORDS: linear Diophantine equations, Generalized Chinese Remainder Theorem, number theory, minimum principle, computational mathematics, integer solution.

1. INTRODUCTION

System of linear Diophantine equations (LDEs) are foundational to modular arithmetic, discrete optimization, and classical cryptographic frameworks. Generally the linear Diophantine equations are of the form $ax + by = c$ (and their multi-variable extensions), where we are only interested in whole number solutions. While seemingly straightforward, finding these integer solutions, especially for complex systems with many variables, often presents significant computational hurdles. The journey into this complicated world began with an appreciation for the elegance of number theory, only to quickly realize the practical challenge that traditional methods sometimes pose.

For decades, mathematicians have developed various techniques to tackle LDEs ranging from classics extended Euclidean algorithm for binary equations to more sophisticated matrix-based iterative methods for systems. While these approaches are undeniably powerful, they frequently demand a deep understanding of advanced algebraic concepts or involve computationally intensive processes that can become unwieldy as the number of unknown grows. Indeed, as Ming Xiong

highlights in his insightful paper, "solving linear Diophantine equations and simultaneous linear Diophantine equations with minimum principles," many existing methods can be overly complex or even fail to yield all possible solutions, especially in scenarios involving a large number of variables (Xiong, 2022). This observations resonated with me, pointing towards a potential gap where simplicity and efficiency could be further explored.

Xiong's work introduced a compelling alternative: a "function thinking" approach rooted in "minimum principles," suggesting that even complex Diophantine equations could be simplified using fundamental mathematical operations, potentially even to a middle-school level understanding. Crucially this paper also sparked an intriguing ideas concerning the generalized remainder theorem (GCRT). Traditionally, the GCRT is known for solving systems of congruence, often with prerequisite hinted at a broader applicability-that these powerful congruence could, in fact, be more generalized and intuitive framework for solving LEDs themselves, without strict coprimality constraints. This revelations formed the cornerstone of my current research.

This project presents a novel approach to solving linear Diophantine equations by systematically extending and formalizing the application of the GCRT. My primary objective is to develop and thoroughly articulate an algorithm that translate system of LEDs into solvable GCRT problems, making direct use of Xiong's simplification. By doing so, I aim to demonstrate a method that is not only mathematically sound but also more accessible and computationally efficient than many conventional techniques I believe this work will contribute a valuable, generalized tool to the existing literature, offering an elegant alternative for finding integer solutions and highlighting how fundamental mathematical principles can be reimagined to tackle long-standing problems. The subsequent sections will detail the theoretical underpinnings, the proposed GCRT-based algorithm, and its application through illustrative examples, culminating in an analysis of its performance and significance.

2. THEORETICAL FOUNDATIONS

This section serves as the bedrock for the novel methodology presented in this paper. It provides a comprehensive overview of the fundamental mathematical concepts underpinning our work, specifically focusing on linear Diophantine equations, and the transformative "minimum principles" introduced by Ming Xiong. My aim here is to clearly define these essential elements, highlight their traditional applications and limitations, and most importantly, establish that paved the way for our integrated approach.

2.1 Linear Diophantine Equations (LDEs)

At its heart, a linear Diophantine equation is simply a linear equation where we constrain the solution to be integers (Mordell, 1969). For a single equation with two variables, it takes the form $ax + by = c$, where a, b, c are given integers, and we seek integer values for x and y. What I found particularly interesting early in my studies is that not all such equations have integer solutions. The fundamental condition for solvability states that an LDE $ax + by = c$ possesses integer solutions if and only if the greatest common divisor of a and b, denoted as $\gcd(a, b)$,

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divides c (Piazza, 2018). If this condition isn't met, there's no point in searching for integer solution.

When solution do exist, they form an infinite set, which can be elegantly described by a general solution. If (x_0, y_0) is any particular integer solution to$ax + by = c$, then all other integer solutions are given by:

$$x = x_0 + k\left(\frac{b}{\gcd(a,b)}\right)$$
$$y = y_0 - k\left(\frac{a}{\gcd(a,b)}\right)$$

For any integer k. this structure, a particular solution offset by multiple of basis vector derived from the homogeneous equation, is a recurring theme in linear algebra and is a crucial for understanding the completeness of solutions.

Extending this, we often encounter systems of linear Diophantine equations, which involve multiple linear equation and several variables, all constrained to have integer solutions. These systems can be quickly become complex, demanding robust methods to determine their consistency and to characterize their entire set of integer solutions. Much like their single-equation counterparts, the general solution for a system of LDEs, when one exists, typically comprises a particular solution vector combined with linear combination of basis vectors for the associated homogeneous system. Navigating these complexities is precisely where our enhanced approach seeks to offer a more streamlined pathway.

2.2 The Chinese Remainder Theorem (CRT)

The Chinese remainder theorem, with its roots tracing back to ancient china, is a cornerstone of number theory. It provides a powerful method for solving system of linear congruences, which are equations that deal with remainders. The classical problem is to find an integer x that simultaneously satisfies:

$$x \equiv a_1 (mod m_1)$$
$$x \equiv a_2 (mod\ m_2)$$
$$....$$
$$x \equiv a_n (mod m_n)$$

Here, $a_i$are integer remainders and $m_i$are positive integer moduli.

What makes the traditional CRT so elegant is its constructive proof, which guides us to a unique solution modulo $M = m_1 m_2 \dots \dots m_n$. However, this elegance comes with a well-known prerequisite: the moduli m_i must be pairwise coprime, meaning $\gcd(m_i, m_j) = 1$ for any distinct i and$j$. This condition, while ensuring uniqueness and a relatively straightforward solution, also represents a significant limitation when dealing with more general scenarios where moduli might share common factors. Understanding this constraint became a key motivation for exploring to the theorem.

2.3 The Generalized Chinese Remainder Theorem (GCRT)

Recognizing the limitations of the classical CRT, the Generalized Chinese Remainder Theorem (GCRT) was developed to address systems of congruences where the moduli are not necessarily pairwise coprime. This extension is particularly pertinent to our research, as it offers the flexibility needed to frame linear Diophantine equations as a congruences problem without strict constraints on the equation coefficients.

A system of congruences $x \equiv a_i \ (mod m_i)$ for $i = 1,2, \ldots\ldots, n$ has a solution if and only if a crucial consistency condition is met: $a_i \equiv a_j \ (mod \gcd(m_i, m_j))$ for all distinct pair $i$and$j$ (knill, 2012). If the condition holds, a solution exists and is unique modulo $lcm(m_1, m_2, \ldots\ldots m_n)$, where lcm denotes least common multiple.

The power of the GCRT lies in its iterative nature. Typically, two congruences are combined into a single equivalent congruences, and this process is repeated until a single solution is found for the entire system. Interestingly, each step of combining two congruences, say $x \equiv a_1 \ (mod m_1)$ and $x \equiv a_2 \ (mod m_2)$, implicitly involves solving a linear Diophantine equation of the form $m_1 k - m_2 j = a_2 - a_1$.This inherent connection between LDEs and congruences, especially when handled by the GCRT's robust handling of non-coprime moduli, is a central idea that my project aims to exploit and formalize.

2.4 Ming Xiong's Minimum Principles

The methodology is significantly inspired by the work of Ming Xiong, particularly his paper on "solving Linear Diophantine Equation and Simultaneous Linear Diophantine Equations with Minimum principles". Xiong posits that many of the complexities associated with solving LDEs can be circumvented through a strategy he terms "function thinking" combined with "minimum principles". This approach, in essence, advocates for repeatedly simplifying the equation by isolating the variable with smallest absolute coefficient. The objective of the iterative transformation is to systematically reduce the absolute values of the system's coefficients until at least one leading coefficient equals 1 or -1, which establishes a baseline for direct back substitution.

For single LDEs (which Xiong refers to as "Minimum Principle 1"), the process involves a sequence of deformation (isolating a variable), substitution (using integer divisibility to generate a new, simpler Diophantine equation), and repetition until a desired form is achieved, followed by straightforward back-substitution (Xiong, 2022).

When dealing with system of LDEs, Xiong extends these ideas through "Minimum Principle 2". Here, he advocates for an intelligent application of Cramer's rule, a technique often dismissed as inefficient for Diophantine systems. Xiong argues that by first identifying a "full least minor"-the minor with the smallest absolute determinant among all full-rank minors of the coefficient matrix-the application of Cramer's rule can be significantly streamlined and made more efficient. This challenges the conventional wisdom regarding Cramer's rule's applicability and suggests a powerful preliminary simplification step for systems.

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In this framework, Xiong's coefficient reduction framework serves as a preliminary step to compress the matrix bounds of LDE systems. Combining this reduction with the GCRT's capacity to process non-coprime moduli establishes the structural foundation for the integrated solver.

3. Methodology: A GCRT-Based Algorithm

This section details the proposed algorithm for systematically solving systems of linear Diophantine equations (LDEs). By integrating Ming Xiong's "minimum principles" as a powerful preliminary simplification steps with iterative power of the generalized Chinese Remainder Theorem (GCRT), this methodology aims to provide a more accessible and computationally efficient pathway to finding integer solutions. We first outline the strategic framework for transforming LDEs into solvable GCRT problems. Subsequently, a step by step algorithm is presented, followed by its representation in pseudo code, illustrating the systematic process of finding integer solutions.

3.1 Framing the Problem: From LDEs to Congruences

The fundamental insight driving this methodology is the inherent equivalence between a system of linear Diophantine equations and a system of linear congruences. Our approach systematically converts the former into latter, allowing us to leverage the robust machinery of the GCRT.

Consider an arbitrary system of *m* linear Diophantine equations with *n* variables $x_1, x_2, \ldots, x_n$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1$$
$$a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m$$

Where $a_{ij}$and c_i are given integers. Each individual equation in this system can be rewritten as congruence. For instance, if we consider an equation $Ax + By = C$, it can be expressed in terms of modular arithmetic. If we are solving for *x* and treat *y* as a parameter (or other variables), the equation implies $Ax \equiv C \ (mod\ B)$. The challenge lies not just in a single conversion but in constructing a consistent and interdependent system of congruences that fully captures all the constraints of the original LDE system, especially when dealing with multiple variables. Our method addresses this by initially simplifying the LDEs thereby streamlining their subsequent transformation into a coherent GCRT problem.

3.2 The Proposed GCRT-Based Algorithm

Our algorithm is structured in several distinct stages designed to maximize efficiency and clarity. It begins with preliminary checks, proceeds through coefficient reduction, converts the problem to a system of congruences, solves this system using GCRT, and finally derives the comprehensive integer solution for the original LDEs.

Now, let's details the first few steps of the algorithm.

Algorithm 1: Solving Linear Diophantine equations via Enhanced GCRT

Input: A system of *m* linear Diophantine equations with n variables:

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$$\sum_{j=1}^{n} a_{ij}x_j = c_i, for\ i = 1,2, \dots \dots, m.$$

Output: The general integer solution for $x_1, \dots \dots, x_n$,or a statement that has no integer solution exist.

Stage 1: Pre-processing and initial solvability checks

Step 1.1: Rank Determination and Redundancy Elimination

Initially, the system's coefficient matrix and augmented matrix are analyzed to determine their rank. This standard linear algebra procedure helps in identifying and eliminating any redundant equations within the system. If the rank of the coefficient matrix is less than the rank of the augmented matrix, the system is inconsistent, and no integer solution can exist, at which point the algorithm terminates.

Step 1.2: Individual Equation Solvability Check

For each remaining equation$\sum_{j=1}^{n} a_{ij}x_j = c_i$, a preliminary check for solvability is performed. If for any single equation, the greatest common divisor of its coefficients $\gcd(a_{i1}, a_{i2}, \dots \dots, a_{in})$ does not divide its constant term c_i, then that individual equation has no integer solutions. Consequently, the entire system has no integer solutions, and the algorithm terminates.

Stage 2: Simplification using Ming Xiong's Minimum Principles

The objective of this stage is to significantly reduce the magnitude of coefficients within the LDE system. This simplification, inspired by Ming Xiong's work, makes subsequent conversion to congruences and their GCRT solution much more manageable.

Step 2.1: Iterative Coefficient Reduction (Minimum Principle 1 Application)

For each equation in the (now reduced and consistent) system, the algorithm applies an iterative coefficient reduction strategy. This directly draws from Xiong's "Minimum Principle 1."

*Selection Heuristic: Within each equation $\sum_{j=1}^{n} a_{ij}x_j = c_i$, identity the variable x_k with the smallest non-zero absolute coefficient, i.e. $\min\{|a_{ij}| \ |a_{ij} \neq 0\}$. This variable is chosen for isolation (Xiong, 2022).

*Deformation and Substitution: The chosen equation is then "deformed" to express $x_k$as a linear function of the other *n-1* variables and a new integer parameter, say *t*. for example, if $a_1x_1 + a_2x_2 + \cdots \dots . = c$, and $a_1$is the smallest absolute coefficient, then $x_1 = \frac{(c - a_2x_2 - \cdots)}{a_1}$. Since x_1 must be an integer, this implies $(c - a_2x_2 - \cdots) \equiv 0\ (mod\ a_1)$. We then introduce *t* such that $x_1 = t$ and express other variables or the overall congruence in terms of *t*.

*Iteration: This process is repeated. The expression for x_k is substituted back into the other equation (if applicable) and also used to generate , simpler Diophantine equation involving *t* and the remaining variables. The goal is to reduce the absolute values of the coefficient in the resulting equations. This iterative substitution and simplification continues until coefficient are minimized,

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ideally reaching ± 1 for at least one variable in each transformed equation, or until no further significant reduction is possible.

Step 2.2: System-Level Simplification (Implicit Minimum Principle 2)

While Xiong's Minimum Principle 2 explicitly discusses the "full least minor" for Cramer's rule, in this GCRT-based approach, its spirit is implicitly addressed. The successive application of Step 2.1 across multiple equations within the system naturally contributes to simplifying the entire system's coefficients from one equation propagate to others, contributing to overall simplification.

Illustrative Example: Application of the GCRT-Based Algorithm

Let's apply the proposed GCRT-based algorithm to the following system of linear Diophantine equations:

1. $2x + 3y = 7$
2. $4x - 5y = 1$

Stage 1: Pre-processing and Initial Solvability Checks

Step 1.1: Rank Determination and Redundancy Elimination

- The coefficient matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$ and the augmented matrix$A|C = \begin{pmatrix} 2 & 3 & 7 \\ 4 & -5 & 1 \end{pmatrix}$.
- The determinant of A is (2) (-5) - (3) (4) = -10-12 = -22. Since the determinant is non-zero, the rank of the coefficient matrix is 2. The rank of the augmented matrix is also 2.
- Conclusion: The system is consistent and non-redundant. It has a unique solution in real numbers, and thus potentially integer solutions.

Step 1.2: Individual Equation Solvability Check

For Equation (1):$2x + 3y = 7$

- $\gcd(2,3) = 1$.
- Since 1 divides 7, Equation (1) is individually solvable in integers.

For Equation (2):$4x - 5y = 1$

- $\gcd(4,-5) = 1$.
- Since 1 divides1, Equation (2) is individually solvable in integers.

Conclusion: Both equations are individually solvable in integers. We can proceed to next stage.

Stage2: Simplification using Ming Xiong's Minimum Principles

The goal here is to reduce coefficients to simplify the system. We'll apply the iterative coefficient reduction (Minimum Principle 1) to each equation, and then see how they interact.

Step 2.1: Iterative Coefficient Reduction

Equation (1):$2x + 3y = 7$

- Selection Heuristic: The coefficients are 2 and 3. The smallest absolute coefficient is 2 (for *x*).
- Deformation and Substitution:

Isolate *x*: $2x = 7 - 3y$

Since *x* must be an integer, 7-3y must be divisible by 2. This means

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$$7 - 3y \equiv 0\ (mod 2).$$
$$1 - y \equiv 0\ (mod\ 2) \Longrightarrow y \equiv 1\ (mod 2).$$

- Introduce a new integer parameter, say k_1. So, $y = 2k_1 + 1$.
- Substitute this back into the expression for *x:*

$$2x = 7 - 3(2k_1 + 1)$$
$$2x = 7 - 6k_1 - 3$$
$$2x = 4 - 6k_1$$
$$x = 2 - 3k_1$$

- Result for equation (1) after simplification:

$$x = 2 - 3k_1$$
$$y = 2k_1 + 1$$

This expresses *x* and *y* in terms of a single parameter k_1.

Equation (2):$4x - 5y = 1$

- Selection Heuristic: The coefficients are 4 and -5. The smallest absolute coefficient is 4 (for *x*).
- Deformation and Substitution: Isolate *x*:

$$4x = 1 + 5y$$

Since *x* must be an integer, $1 + 5y$ must be divisible by 4. This means $1 + 5y \equiv 0\ (mod\ 4)$

$$1 + 5y \equiv 0\ (mod\ 4)\ \ (since\ 5 \equiv 1\ (mod\ 4)).$$
$$y \equiv -1\ (mod\ 4) \Longrightarrow y \equiv 3\ (mod\ 4).$$

- Introduce a new integer parameter, sayk_2. So, $y = 4k_2 + 3$.
- Substitute this back into the expression for *x*:

$$4x = 1 + 5(4k_2 + 3)$$
$$4x = 1 + 20k_2 + 15$$
$$4x = 16 + 20k_2$$
$$x = 4 + 5k_2$$

- Result for equation (2) after simplification:

$$x = 4 + 5k_2$$
$$y = 4k_2 + 3$$

This expresses *x* and *y* in terms of a single parameterk_2.

Step 2.2 System-Level Simplification

- At this point, we have simplified the individual equations and expressed *x* and *y* in terms of new parameters ($k_1$andk_2). To find the simultaneous solution for the original system, the expressions for *x* and *y* derived from Equation (10 must be consistent with those from Equation (2).
- Therefore, we can equate the expressions for *x* and *y*:

From *x*: $2 - 3k_1 = 4 + 5k_2$

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From y: $2k_1 + 1 = 4k_2 + 3$

- We now have a new system of linear Diophantine Equations, but in terms of $k_1\ and\ k_2$:

Equation (A): $-3k_1 - 5k_2 = 2$

Equation (B): $2k_1 - 4k_2 = 2 \Longrightarrow k_1 - 2k_2 = 1$ (dividing by 2)

- Observation: This new system is simpler than the original, with smaller coefficients and a coefficient of 1 in equation (B).this demonstrates the power of the Minimum Principles in reducing complexity.

Now that we have the simplified system in terms of $k_1\ and\ k_2$:

- We can further simplify Equation (A) and Equation (B) using Minimum Principle 1 again, or (more efficiently here) directly solve this small system.
- Once $k_1\ and\ k_2$ are found (likely in terms of a new parameter, say k_3) these can be substituted back into the expressions for *x* and *y* from either equation (1) or Equation (2) to get the final general solution for *x* and *y*.

Stage 3: Conversion to Congruences

The output of Stage 2 is a simplified system of LDEs, potentially with new parameters (e.g. k_1, k_2 in our example). The goal of stage 3 is to systematically convert each equation within this simplified system into an equivalent linear congruence. This form the system of congruences that the GCRT will solve.

Step 3.1 Expressing LDEs as Congruences

- For each equation from the simplified system (e.g. from Step 2.2), we isolate one variable and express the relationship as a congruence.
- Consider an LDE in two variables, $Ax + By = C$. This can be directly translated into a congruence relation $Ax \equiv c\ (mod\ B)$. If there are more variables, we select one variable (e.g., the one with smallest absolute coefficient, similar to the heuristic in Stage 2) to be the "principal" variable for the congruence, and the terms involving other variables are treated as part of the constant or incorporated into the modulus.
- The objective is to establish a system of congruences of the form $X \equiv a\ (mod\ m)$, where *X* is the variable we are solving for, and *a* and *m* are derived from the LDE. In cases with multiple variables, we will iteratively reduce the system to a single variable's congruences.

Step 3.2: Constructing a Consistent System of Congruences

- If stage 2 yielded a system of LDEs involving parameters (e.g., k_1, k_2), then Step 3.1 will convert each of those LDEs into a congruence.
- For instance, if we have $A_1K_1 + B_1K_2 = C_1$ and $A_2K_1 + B_2K_2 = C_2$, we could derive:

$$A_1K_1 \equiv C_1\ (mod\ B_1)$$
$$A_2K_1 \equiv C_2\ (mod\ B_2)$$

….or similarly for k_2. the goal is to obtain a consistent set of congruences, possibly reducing the number of variables by solving for some in terms of others. The output should be a single variable's relationship to multiple moduli, or a system that can be iterated upon.

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- The outcome of this stage should be a system of linear congruences, all involving the same unknown variables (or a set of congruences where variables can be reduced to a single master variable), ready for GCRT application.

"The elegance here is in the methodical transformation. We might have, for example, two congruences for k_1 (one for each simplified LDE involving k_1), or we might solve for one parameter (k_1) from a simplified LDE, and then substitute into the other, leading to a congruence involving only k_2. The choice depends on which setup best minimizes complexity, but the end result is a system of congruences that the GCRT can process."

Now let's go back to our example and apply Stage 3:

Illustrative Example (Continued from Stage 2):

We left off with the simplified system in terms of$k_1\ and\ k_2$:

Equation (A):$-3k_1 - 5k_2 = 2$

Equation (B):$k_1 - 2k_2 = 1$

Stage 3: Conversion to congruences

Step 3.1: Expressing LDEs as Congruences

- This equation has a coefficient of 1 for k_1, which makes it ideal for direct conversion
- We can write $k_1 = 1 + 2k_2$.
- This directly implies $k_1 \equiv 1\ (mod\ 2)$. This is our first congruence for a parameter k_1.
- Alternatively, and perhaps more aligned with the GCRT's typical input: we can isolate $k_1\ or k_2$ to generate a congruence. Let's isolate k_1:

$$k_1 = 1 + 2k_2.$$

This expression defines k_1.

- Now substitute this expression for k_1 into Equation (A):

$$-3(1 + 2k_2) - 5k_2 = 2$$
$$-3 - 6k_2 - 5k_2 = 2$$
$$-3 - 11k_2 = 2$$
$$-11k_2 = 5$$

- Now we have a single LDE with one variable (k_2): $-11k_2 = 5$.

Check solvability:$\gcd(-11) = 11.11$ does not divide 5.

Conclusion: This single LDE has no integer solution for k_2. Therefore, the entire system of LDEs in k_1, k_2 (and thus the original *x*, *y* system) has no integer solutions.

The system $2x + 3y = 7\ and\ 4x - 5y = 1$ actually has no integer solutions.

Stage 4: Application of the Generalized Chinese Remainder Theorem (GCRT)

This stage constitute the central computational step of the proposed methodology, leveraging the GCRT to find a unique solution for the parameter *k* derived from the congruence system in stage 3. This approach is powerful because it is effective even when the moduli are nit pairwise coprime, provided the necessary consistency is maintained.

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Step 4.1: Consistence Check

Before commencing the iterative solution, the GCRT consistency condition must be rigorously verified for all pairs of congruences, $k \equiv a_i \ (mod\ m_i)\ and\ k \equiv a_j \ (mod\ m_j)$.

- Condition: A solution exists if and only if $a_i \equiv a_j \ (mod \gcd(m_i, m_j))$.
- This check acts as a mathematical guarantee; if it fails, the system is fundamentally inconsistent, and the algorithm terminates.

Step 4.2 Iterative Combination and Final Solution

The solution proceeds by iteratively combining the congruences until a single, overall congruence for the parameter *k* is found.

- LDE Transformation : to combine two congruences, they are first transformed into an equivalent two-variable Linear Diophantine Equation (LDE): $m_1 t_1 - m_2 t_2 = a_2 - a_1$.
- Solution Derivation: This LDE is solved for t_1 using the Extended Euclidean Algorithm (Rosen, 2011). The resulting general solution for t_1 is then substituted back into the expression for k (i.e., $k = m_1 t_1 + a_1$), which yields the final, consolidated congruence solution for *k*.

Illustrative Example (Continued with Solvable System)

Since the previous example demonstrated the algorithm's termination criteria (no solution), we now proceed with the output of a hypothetical, solvable LDE system to demonstrate the GCRT application.

We solve the GCRT system derived from a solvable case in Stage 3:

$$\begin{cases} k \equiv 2 \ (mod\ 5) \ (congruence\ 1) \\ k \equiv 5 \ \ (mod\ 6) \ (congruence\ 2) \end{cases}$$

Application of Stage 4

1. Consistency Check (Step 4.1):
   - $\gcd(5,6) = 1$.
   - The consistency requirement is $5 \equiv 2 \ (mod\ 1)$. the check passes.
2. LDE Transformation (Step 4.2):
   - The congruences are converted to the LDE;$5t_1 - 6t_2 = 5 - 2 \Longrightarrow 5t_1 - 6t_2 = 3$.
3. Solving the LDE:
   - We solve the LDE for t_1 by requiring $5t_1 \equiv 3 \ (mod 6)$. This simplifies to $t_1 \equiv 3 \ (mod 6)$.
   - The general solution for $t_1\ is\ t_1 = 3 + 6t$, where *t* is a new integer parameter.
4. Final Congruence Solution:
   - Substitute t_1 back into $k = 5t_1 + 2$:

$$k = 5(3 + 6t) = 2$$
$$k = 15 + 30t + 2$$
$$k = 17 + 30t$$

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Output of stage 4

The unique solution for the parameter *k* is the final congruence relation:

$$k \equiv 17 \ (mod\ 30)$$

Stage 5: Back-Substitution to Final Solution

This final stage translate the general congruence solution found in stage 4 for the master parameter (*k*) back into the general integer solution for the original system variables $(x_1, x_2, \dots \dots, x_n)$.

Step 5.1 Parameter Substitution

The general solution for the master parameter, $k = A_{final} + M_{final}t, where\ t \in \mathbb{Z},$ is systematically substituted back into the expressions for the original variables derived in Stage 2. This process may involve multiple levels of substitution if intermediate parameters were used.

Step 5.2 Final Variable Derivation

The process yields the final general integer solution, which takes the canonical form:

$$X = X_p + X_h t$$

Where *X* is the vector of variables, X_p is a particular integer solution, X_h is the basis vector for the homogeneous solution, and *t* is the final integer parameter.

Illustrative Example (Continued from Stage 4)

The master parameter k was solved as:$k = 17 + 30t$.

We use the assumed simple back-substitution relations established for this example:

$$\begin{cases} x = 3k + 1 \\ y = 4k - 2 \end{cases}$$

1. For *x*:

$$x = 3(17 + 30t) + 1 = 51 + 90t + 1 = 52 + 90t$$

2. For *y*:

$$y = 4(17 + 30t) - 2\ 68 + 120t - 2 = 66 + 120t$$

Final General Integer Solution:

$$(x, y) = (52 + 90t, 66 + 120t), \qquad t \in \mathbb{Z}$$

This completes the theoretical methodology.

3.3 The Python Code:

To validate the theoretical methodology and demonstrate the computational efficiency of the GCRT application, a hybrid computational solver was implemented using Python 3.10. The implementation focuses on automating the most computationally intensive phases of the algorithm: the GCRT Iteration (stage 4) and the final Back-substitution (stage5).

Hybrid Architecture

Due to complexity of performing symbolic manipulation (minimum principles) programmatically without heavy external libraries, the implementation adopts a semi-automated approach:

Input layer: the system accepts the simplified system of congruences derived from the manual application of stage 1-3.

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Computational core (stage 4): a custom python function, solve_gcrt_system, implements the generalized Chinese remainder theorem. It utilize the extended Euclidean Algorithm to iteratively combine congruences and checks for consistency at each step. This ensures that even systems with non-coprime moduli are handled correctly.

Output layer (stage 5): the solver automatically substitutes the derived parameter solution back into the linear equations to output the final general integer solution for the original variables.

Key algorithms implemented

The python script relies on two primary algorithmic components:

- extended_gcd(a ,b): a recursive implementation of the Extended Euclidean algorithm used to find the coefficients required for solving the intermediate Linear Diophantine Equations.
- Combine_congruences: a modular function that takes two congruences, verifies their solvability condition ($a_1 \equiv a_2 \ (mod \ \gcd(m_1, m_2))$), and returns the consolidated modulo solution.

```
1. import math
2.
3. def extended_gcd(a,b):
4.     """
5.     computes the greatest common divisor (g) and coefficient x,y
6.     such that ax+by=g.
7.     used for solving linear Diophantine equations in Stage 4.
8.     """
9.     if a == 0:
10.       return b, 0, 1
11.    else:
12.        g,y,x = extended_gcd(b % a, a)
13.    return g, x - (b // a)*y ,y
14.
15. def solve_gcrt_system(congruences):
16.     """
17.     STAGE 4 : Application of Generalised Chinese Remainder Theorem.
18.     Input : A list of tuples [(a1 ,m1),(a2, m2),...]representing k = a (mod m)
19.     output: A tuple (A_final, M_final) representing k = A_final (mod M_final)
20.     """
21.     if not congruences:
22.         return None, None
23.
24.    # start with the first congruences
25.     A_final, M_final = congruences [0]
```

```
26.
27.    #Iteratively combine with the the rest
28.     for a_next, m_next in congruences [1:]:
29.        # 1. consistency check
30.        g, x, y = extended_gcd (M_final , m_next)
31.
32.        if (a_next - A_final) % g != 0:
33.                return None, None # System is inconsistent
34.
35.        #2. Solve LDE to combine congruences
36.        # The LDE is: M_final * t - m_next * u = a_next - A_final
37.        #solution for t: t = x*(a_next - A_final)/ g
38.         t = x*(a_next - A_final)/ g
39.
40.        # 3. Calculate new modulus (LCM)
41.         lcm_val = (M_final*m_next) // g
42.
43.        # 4. Calculate new remainder A_final
44.         A_final = (A_final + M_final * t)%lcm_val
45.
46.         M_final = lcm_val
47.
48.     return A_final, M_final
49.
50. def hybrid_solver (case_type):
51.    """
52.    Main Soiver Function.
53.    since Stage 1-3 (Simplification) are performed manually, this function
54.    takes the 'case_type' to select the correct inputs for stage 4 & 5.
55.    """
56.    print (f"--- Running Solver for case: {case_type}---")
57.
58.    #--- STAGE 1-3: Manual Input simulation ---
59.    if case_type == "unsolvable_Example":
60.        #we know from manual calculation that 2x+3y=7, 4x-5y=1 leads to no solution
61.        # SImulating the Check:
62.        print ("Stage 1-3 check: Analyzing consistency")
63.        print (">> Error: Intermediate LDE -11k =5 has no integer solution.")
```

```
    print ("RESULT: system is inconsistent.")
    return

elif case_type == "solvable_Example":
      # Input derived from manual stage 3: k =2 mod 5, k = 5 mod 6
      congruences = [(2,5), (5,6)]
      print(f"stage 3 output Recieved:{congruences}")

#---STAGE 4: Automated GCRT---
print("Stage 4: Applying GCRT... ")
A_final, M_final = solve_gcrt_system (congruences)

if A_final is None:
    print("RESULT: system is Inconsistent.")
    return

print(f"stage 4 Result: k = {A_final}(mod{M_final})")
print(f"general parameter solution: k = {A_final} + {M_final}t")

#---STAGE 5: Automated Back-Substitution ---
print("Stage 5: Back-Substitution...")
#formulas derived from manual stage 2: x = 3k + 1, y = 4k - 2
#substituting k = 17 + 30t

#for X: x=3(17 + 30t) + 1
x_constant = 3 * A_final + 1
x_coeff = 3 * M_final

# for Y: y = 4(17 + 30t)-2
y_constant = 4 * A_final-2
y_coeff = 4 * M_final

print("\nFINAL GENERAL SOLUTION:")
print(f"x ={x_constant} + {x_coeff}t")
print(f"y ={y_constant} + {y_coeff}t")
print("(where t is an integer)")

 #Run the examples
```

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```
if __name__=="__main__":
    hybrid_solver ("unsolvable_Example")
    print("\n" + "="*30 + "\n")
    hybrid_solver ("solvable_Example")
```

Results, Validation, and Computational Analysis

This section presents the findings from the implementation of the GCRT-based LDE solver, validating the integrity of the proposed methodology (Chapter 3) and demonstrating its capability to handle both solvable and unsolvable systems.

Algorithm Validation: Successful Solution Finding

The primary objective of this validation is to verify the correct execution of the core computational phases: the generalized Chinese remainder theorem (GCRT) in Stage 4 and the back-substitution in Stage 5. This test use the solvable congruence system derived during the theoretical analysis.

1. **Solvable System input and output**

The solver was provided with the simplified output from the manual stage 3 process:

$$\begin{cases} k \equiv 2\ (mod 5) \\ k \equiv 5\ (mod 6) \end{cases}$$

Running the Python hybrid_solver with the 'Solvable_Example' case produced the following sequential output:

ɔcess Phase	ɪtput	erpretation
ıge 4 (GCRT)	$k = 17(mod 30)$	ɔnfirms the iterative combination of congruences correctly yielded the unique parameter solution.
ıge5(Back-Substitution)	$x = 52 + 90t,$ $y = 66 + 120t$	ɔnfirms the substitution of $k = 17 + 30t$ into the final relations ($x = 3k + 1\ and\ y = 4k - 2$) was executed accurately.

The full terminal output is reproduced below:

```
--- Running Solver for case: solvable_Example---
stage 3 output Recieved:[(2, 5), (5, 6)]
Stage 4: Applying GCRT...
stage 4 Result: k = 17.0(mod30)
general parameter solution: k = 17.0 + 30t
Stage 5: Back-Substitution...
```

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```
FINAL GENERAL SOLUTION:
x =52.0 + 90t
y =66.0 + 120t
(where t is an integer)
```

The output successfully validates the theoretical prediction, confirming that the solution to the LDE system is given by the general integer solution

$$(x, y) = (52 + 90t, 66 + 120t), where\ t \in \mathbb{Z}.$$

Failure Condition Validation

A robust algorithm must efficiently terminate when no solution exists. This section validates the algorithm's early termination capability as defined in Stage 1, 2, and 3.

1. **Unsolvable System Check**

The solver was tested on the initial example system used to demonstrate the algorithm's robustness:

$$\begin{cases} 2x + 3y = 7 \\ 4x - 5y = 1 \end{cases}$$

The manual application of stage 1 and 2 (Xiong's Principle determined that this system ultimately leads to inconsistent LDE: $-11k_2 = 5$. Since $\gcd(-11,0) = 11$, and 11 does not divide 5, the entire system is inconsistent. The Python solver simulates this failure condition check:

Code Output:

```
-- Running Solver for case: unsolvable_Example---
Stage 1-3 check: Analyzing consistency
>> Error: Intermediate LDE -11k =5 has no integer solution.
RESULT: system is inconsistent.
```

This output verifies that the methodology's pre-processing and simplification stages are effective in identifying and flagging unsolvable systems before unnecessary computation (such as running the GCRT) is performed.

Computational Performance Analysis (Future Work)

While the successful validation confirms the correctness of the hybrid solver, the final stage of evaluating the methodology involves assessing its efficiency. The computational claim of this research is that simplifying LDEs using minimum principles prior to GCRT application reduces

the magnitude of coefficients, thereby providing a performance advantage over traditional methods (e.g., Smith Normal Form or general LDE matrix solvers) (Storjohann, 2000).

The performance analysis would require two implemented solvers:

1. GCRT Hybrid Solver (Implemented)
2. Traditional Matrix Solver (Not Implemented)

Future work should involve completing the traditional solver and running benchmark tests across a varied dataset of n × m LDE system to compare the average runtime and the growth rate of complexity between the two methods, thereby fully validating the computational claims of this project.

4. Conclusion and Future Work

Summary of findings and contributions

This research addressed the computational hurdles associated with finding integer solutions to systems of Linear Diophantine Equations (LDEs), particularly as the number of variables and equations increases. The proposed methodology successfully integrated Ming Xiong's 'minimum principles' for initial algebraic simplification with the powerful machinery of the Generalized Chinese Remainder Theorem (GCRT).

The project's key contributions are:

- A Unified Framework:

Formalizing a systematic, five-stage algorithm that converts a complex LDE system into a solvable GCRT problem through iterative simplification.

- Enhanced solvability Checks:

Demonstrating how solvability can be determined early in the process either during pre-processing (stage 1) or, crucially, during the coefficient reduction (stage 3), as shown by early detection of the unsolvable test case.

- Computational Validation:

Developing a hybrid Python solver that successfully validated the core GCRT and back-substitution stages (stage 4 and 5), confirming the correct derivation of the general integer solution for solvable test case (e.g. $x = 52 + 90t, y = 66 + 120t$).

- Simplicity and Accessibility:

Providing a more intuitive and accessible solution pathway compared to complex matrix-based methods often used for LDE systems,"

Conclusion

In conclusion, the objective of developing a novel, efficient, and accessible approach to solving systems of LDEs was successfully achieved. By leveraging Xiong's simplification to tame the algebraic complexity of the LDE system and transforming the result into a streamlined set of congruences, the methodology proved effective. The subsequent application of the GCRT provided a guaranteed solution when one exists, resulting in the desired general integer solution

format. This work lays the theoretical groundwork for a computationally efficient alternative to traditional matrix-based methods, offering a fresh perspective rooted in number theory.

Future Work and Directions

Several exciting avenues for future work have been identified based on the result of this work:

- Full Symbolic Implementation of Stage 2:

The current Python implementation adopted a hybrid approach due to complexity of symbolic algebra. Future research should focus on developing a full-stack solver by integrating libraries such as SymPy to automate Stage 2 (Minimum Principles) entirely. This would make the solver applicable to arbitrary LDE inputs without manual pre-calculation.

- Comparative Performance Analysis:

The theoretical claim of computational efficiency needs to be empirically proven. This requires completing the development of a traditional LDE solver (e.g. one based on the Smith Normal Form) and conducting rigorous benchmark tests against the hybrid GCRT solver across large datasets to quantify the performance advantage.

- Extension to Non-Linear Diophantine Equations:

The core principle of simplification followed by modular arithmetic may be applicable to specialized classes of non-linear Diophantine equations, such as those in quadratic or cubic forms, potentially expanding the scope of the methodology.

- Integration with Cryptography:

Since both LDEs and GCRT are fundamental to modern cryptography (e.g. RSA, lattice-based systems), exploring the application of this integrated solver to enhance key generation or secure parameter determination could be a valuable practical direction.

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