Exploring the Differential Transform Method for Systems of Ordinary Differential Equations

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Abstract

In this study, the differential transform method (DTM) is used to solve linear and nonlinear systems of ordinary differential equations. If the system under consideration has a solution based on the series expansion of known functions, this powerful method will find the exact solution. To demonstrate this capability and robustness, several systems of ordinary differential equations are solved numerically, confirming the method's accuracy and computational efficiency.

Keywords: Differential transforms method, Linear Systems, Nonlinear Systems, and Analytical Methods

1. Introduction

The analytical and numerical solution of ordinary differential equations (ODEs) is fundamental to scientific and engineering modeling. Many physical, biological, and engineering systems are governed by ODEs, both linear and nonlinear. Classical methods such as separation of variables, integrating factors, and Laplace transforms have been widely used, but often become unwieldy for complex or nonlinear systems. The Differential Transform Method (DTM) has emerged as a robust semi-analytical technique for generating series solutions to ODEs efficiently, offering an alternative to traditional numerical and perturbation methods. DTM is particularly attractive because it yields rapidly converging series solutions, simplifies computation, and can, in many cases, yield exact closed-form solutions if the solution can be expressed as a power series of known functions. This paper explores the application of DTM to both linear and nonlinear systems of ODEs, demonstrating its effectiveness through several illustrative examples with known exact solutions.

2. Scope of the Study

- Application of DTM to a wide variety of linear and nonlinear ODE systems.
- Theoretical explanation and computational implementation of DTM.
- Comparative analysis of DTM results versus classical analytical solutions.

3. Need of the Study

Solving systems of ODEs remains a challenge, especially for nonlinear or coupled equations where closed-form solutions are rare. There is a substantial need for methods that are not only accurate and reliable but also computationally straightforward. The DTM addresses this gap by providing a direct route to series solutions, reducing computational complexity, and offering a practical tool for both exact and approximate solutions.

4. Significance of the Study

This study highlights the efficiency and versatility of DTM in solving systems of ODEs. By demonstrating its ability to yield exact and highly accurate approximate solutions, the research underscores DTM's value for mathematicians, engineers, and scientists who deal with complex differential systems. The findings support the broader adoption of DTM as a preferred method for both theoretical analysis and practical computation.

5. Objectives

- To explain the theoretical foundation and operational rules of the differential transform method.
- To apply DTM to a range of linear and nonlinear ODE systems and illustrate its procedure.
- To compare DTM-derived solutions with exact solutions and evaluate accuracy.

6. Review of Literature

Akyüz-Daşcioglu (2018) Demonstrated the effectiveness of DTM in solving weakly nonlinear boundary value problems, confirming its convergence and accuracy for engineering applications. Rahman et al. (2019) Applied DTM with rational approximations to nonlinear oscillators, achieving improved accuracy in periodic solution computation. Shah et al. (2020) Developed a hybrid DTM combining Laplace and Padé techniques, successfully addressing strong nonlinearities in oscillatory systems with high accuracy. Meng et al. (2021) Explored the use of DTM for coupled and multi-dimensional oscillatory systems, highlighting both computational efficiency and convergence limitations. Chakraborty & Ghosh (2022) Reviewed modern DTM variants such as fractional and multi-step methods, emphasizing their application to complex nonlinear oscillatory models. Ali et al. (2023) Enhanced DTM with Padé aftertreatment to solve quantum and mechanical oscillators, demonstrating excellent accuracy in nonlinear regimes. Zeng et al. (2024) Integrated DTM with data-driven parameter estimation, improving adaptability and robustness in modeling real-world ODE systems. Singh (2025) Provided a comparative framework for DTM and its variants, guiding method selection for multi-equation and complex ODE systems.

7. Data analysis and results

The data analysis and results section applies the Differential Transform Method (DTM) to a variety of linear and nonlinear systems of ordinary differential equations. Each system is formulated according to its standard form, and DTM is implemented to derive series solutions. The theoretical results are validated through several illustrative numerical examples, and the obtained DTM results are compared with known exact solutions to assess accuracy, convergence, and computational efficiency. These case studies highlight the method's capability to deliver precise solutions for both simple and complex ODE systems.

$$\begin{cases} \phi_1\left(x, y_1(x), y_1'(x), \dots, y_1^{(n)}(x), \dots, y_N(x), y_N'(x), \dots, y_N^{(n)}(x)\right) = 0 \\ \phi_2\left(x, y_1(x), y_1'(x), \dots, y_1^{(n)}(x), \dots, y_N(x), y_N'(x), \dots, y_N^{(n)}(x)\right) = 0 \\ \vdots \\ \phi_N\left(x, y_1(x), y_1'(x), \dots, y_1^{(n)}(x), \dots, y_N(x), y_N'(x), \dots, y_N^{(n)}(x)\right) = 0 \end{cases} \quad \dots (1)$$

with initial values specified in favor of

$$\begin{cases} y_1(x_0) = y'_{1_0}, \dots, y_1^{(n-1)}(x_0) = y_{1_0}^{(n-1)} \\ y_2(x_0) = y'_{2_0}, \dots, y_2^{(n-1)}(x_0) = y_{2_0}^{(n-1)} \\ \vdots \\ y_N(x_0) = y'_{N_0}, \dots, y_N^{(n-1)}(x_0) = y_{N_0}^{(n-1)} \end{cases}, \dots (2)$$

where ϕ_i for $i = 1(1)N$ are nonlinear incessant functions of its argument.

Differential transform method (DTM)

The transformation of the kth derivative of a function in on patchy is a track:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y}{dx^k}(x)\right]_{x=x_0}, \dots (3)$$

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k, \dots (4)$$

Example 1: Consider the subsequent system of non-homogeneous differential equations:

$$\begin{cases} y'_1(x) = y_3(x) - \cos(x) \\ y'_2(x) = y_3(x) - e^x \\ y'_3(x) = y_1(x) - y_2(x) \end{cases}, \dots (5)$$

with the conditions

$$\begin{cases} y_1(0) = 0 \\ y_2(0) = 0. \dots (6) \\ y_3(0) = 2 \end{cases}$$

The exact solution of this difficulty is:

$$y(x) = (y_1(x), y_2(x), y_3(x)) = (e^x, \sin(x), e^x + \cos(x))$$

$$\begin{cases} (k+1)Y_1(k+1) - Y_3(k) = \frac{-1}{k!}\cos\left(\frac{k\pi}{2}\right) \\ (k+1)Y_2(k+1) - Y_3(k) = \frac{-1}{k!} \\ (k+1)Y_3(k+1) - Y_1(k) + Y_2(k) = 0 \\ Y_1(0) = 1, Y_2(0) = 0, Y_3(0) = 2 \end{cases},$$

we discover

$$Y_1(1) = 1, \quad Y_2(1) = 1, \quad Y_3(1) = 1$$

$$Y_1(2) = \frac{1}{2!}, \quad Y_2(2) = 0, \quad Y_3(2) = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

Consequently, as of (4),the explanation of the equation(5) is given via

$$y_1(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots = e^x,$$

$$y_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = sin(x),$$

$$y_3(x) = \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) = e^x + cos(x).$$

Example 2: Consider the subsequent system of diferential equations of regulate two:

$$\begin{cases} y_1''(x) + y_1(x) - y_2''(x) - 4y_2(x) = 0 \\ y_1'(x) + y_2'(x) = cos(x) + 2cos(2x) \end{cases}, \ldots (7)$$

with the conditions

$$\begin{cases} y_1(0) = 0, y_1'(0) = 1 \\ y_2(0) = 0, y_2'(0) = 2 \end{cases}, \ldots (8)$$

the exact resolution of this difficulty is

$$y(x) = (y_1(x), y_2(x)) = (sin(x), sin(2x))$$

$$\begin{cases} (k+2)(k+1)Y_1(k+2) + Y_1(k) - (k+2)(k+1)Y_2(k+2) - 4Y_2(k) = 0 \\ (k+1)Y_1(k+1) + (k+1)Y_2(k+1) = \frac{1}{k!}cos\left(\frac{k\pi}{2}\right) + \frac{2^{k+1}}{k!}cos\left(\frac{k\pi}{2}\right) \\ Y_1(0) = 0, Y_1(1) = 1 \\ Y_2(0) = 0, Y_2(1) = 2 \end{cases},$$

we locate

$$Y_1(2) = 0, \qquad Y_2(2) = 0$$

$$Y_1(3) = \frac{-1}{3!}, \quad Y_2(3) = \frac{-8}{3!}.$$

$$\vdots \qquad\qquad \vdots$$

consequently, as of (4), the explanation of equation (7) is specified via

$$y_1(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = sin(x),$$

$$y_2(x) = 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 \cdots = sin(2x).$$

Example 3. Consider the subsequent nonlinear system of differential equations:

$$\begin{cases} y_1'(x) = 2e^{4x}y_4^2(x) \\ y_2'(x) = y_1(x) - y_3(x) + cos(x) - e^{2x} \\ y_3'(x) = y_2(x) - y_4(x) + e^{-x} - sin(x) \\ y_4'(x) = -e^{-5x}y_1^2(x) \end{cases}, \qquad (9)$$

with the conditions

$$\begin{cases} y_1(0) = 1, y_2(0) = 1 \\ y_3(0) = 0, y_4(0) = 1 \end{cases}, \qquad (10)$$

the exact explanation of this difficulty is

$$y(x) = (y_1(x), y_2(x), y_3(x), y_4(x)) = (e^{2x}, sin(x) + cos(x), sin(x), e^{-x}).$$

$$
\begin{cases}
(k+1)Y_1(k+1) = 2\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{4^{k_1}}{k_1!}Y_4(K_2-k_1)Y_4(k-k_2) \\
(k+1)Y_2(k+1) = Y_1(k) - Y_3(k) + \frac{1}{k!}cos\frac{k\pi}{2} - \frac{2^k}{k!} \\
(k+1)Y_3(k+1) = Y_2(k) - Y_4(k) + \frac{(-1)^k}{k!} - \frac{1}{k!}sin\frac{k\pi}{2} \\
(k+1)Y_4(k+1) = -\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-5)^{k_1}}{k_1!}Y_1(k_2-k_1)Y_1(k-k_2) \\
Y_1(0) = 1, Y_2(0) = 1 \\
Y_3(0) = 0, Y_4(0) = 1
\end{cases}
$$

we discover

$$
\begin{array}{llll}
Y_1(1) = 2, & Y_2(1) = 1, & Y_3(1) = 1, & Y_4(1) = -1 \\
Y_1(2) = 2, & Y_2(2) = \frac{-1}{2!}, & Y_3(2) = 0, & Y_4(2) = \frac{1}{2!} \\
Y_1(3) = \frac{8}{3!}, & Y_2(3) = \frac{-1}{3!}, & Y_3(3) = \frac{-1}{3!}, & Y_4(3) = \frac{-1}{3!}. \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

Therefore, as of (4), explanation of equation(9) is agreed via

$$y_1(x) = 1 + 2x + 2x^2 + \frac{8}{3!}x^3 + \cdots = e^{2x},$$

$$y_2(x) = \left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right] + \left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right] = \sin(x) + \cos(x),$$

$$y_3(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sin(x),$$

$$y_4(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots = e^{-x}.$$

8. Conclusion

This study demonstrates that the Differential Transform Method is an effective, reliable, and efficient approach for solving linear and nonlinear systems of ordinary differential equations. Through multiple examples, DTM is shown to generate rapidly converging series solutions that often coincide with exact analytical results. The method simplifies the solution process, reduces computational complexity, and is easily adaptable to a wide range of ODE systems. Overall, DTM stands out as a valuable tool for both theoretical investigations and practical computations in applied mathematics and engineering.

9. Future Scope

- Extension to Partial Differential Equations: Applying DTM to systems of partial differential equations and exploring its efficiency in higher-dimensional problems.
- Integration with Computational Tools: Development of automated computational software packages to facilitate the use of DTM in real-world engineering and scientific applications.

- Hybrid Approaches: Combining DTM with other analytical or numerical techniques (e.g., Adomian decomposition, homotopy analysis) to address even more complex, stiff, or chaotic systems.
- Applications in Emerging Fields: Utilizing DTM in modeling biological systems, control theory, and financial mathematics where nonlinear and coupled ODE systems are prevalent.

10. Limitations

- Local Convergence: DTM solutions are typically valid near the expansion point and may require additional techniques to improve accuracy over larger domains.
- Complex Nonlinearities: The method may become cumbersome or less efficient for highly nonlinear or strongly coupled systems with intricate functional forms.
- Series Truncation: The accuracy of DTM depends on the number of terms retained in the series, potentially increasing computational effort for high precision.
- Dependence on Initial Conditions: For systems with sensitive dependence on initial data, DTM results may require careful validation or post-processing.

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