



Improving the Differential Transform Method for Nonlinear Oscillatory Systems: Theory, Algorithms, and Applications

Sanju Kumari¹; Dr. Ritu Sindhu²

¹Research Scholar, Department of Mathematics, Shri Jagdishprasad Jhabarmal Tibrewala University, Jhunjhunu, Rajasthan, India

²Assistant Professor and Research Guide, Department of Mathematics, Shri Jagdishprasad Jhabarmal Tibrewala University, Jhunjhunu, Rajasthan, India

Abstract

This study looks into the use of the Differential Transform Method (DTM) to analyze linear, weakly nonlinear and nonlinear oscillatory differential equation systems. Recognizing that standard DTM solutions may lack the inherent periodic nature of oscillatory systems, we propose an improved method that uses Padé approximants and Laplace transforms to recover periodicity and extend convergence domains. The methodology is systematically described and validated using representative examples to demonstrate its efficacy, increased accuracy, and computational efficiency. The findings indicate that DTM, particularly when combined with aftertreatment techniques, is a powerful and practical tool for solving a wide range of differential equations in mathematical physics and engineering.

Keywords: Differential Transform Method, Padé Approximant, Nonlinear Oscillatory Systems, Linear Differential Equations, Laplace Transform, Series Acceleration

1. Introduction

The solution of differential equations is central to the mathematical modeling of natural phenomena, technological processes, and engineered systems across diverse fields such as physics, biology, engineering, economics, and environmental science. Differential equations, whether ordinary or partial, serve as mathematical representations for the dynamic evolution of systems over time or space, encapsulating relationships among variables and their rates of change. Their analytical and numerical treatment underpins progress in scientific discovery, technological innovation, and industrial design.

In recent decades, the growing complexity of real-world systems has led to a surge in interest in both semi-analytical and computational methods for solving differential equations. While classical methods such as separation of variables, variation of parameters, and Laplace transforms remain foundational, they are often limited to linear or weakly nonlinear systems with simple boundary or initial conditions. The demand for efficient, accurate, and broadly applicable solution techniques has thus driven the development of new approaches capable of addressing nonlinearity, strong coupling, and oscillatory dynamics.

One such approach that has gained prominence is the Differential Transform Method (DTM), first introduced by Zhou in 1986 for solving electrical circuit equations. DTM is a semi-analytical method that constructs an approximate solution in the form of a truncated power series, derived systematically from the governing differential equation and initial

conditions. Unlike the direct Taylor series expansion, DTM leverages operational properties that transform differential and algebraic operations into manipulations of series coefficients, significantly simplifying the process for both linear and nonlinear differential equations.

DTM stands out for its algorithmic simplicity, computational efficiency, and versatility. It does not require linearization, small parameter assumptions, or perturbation expansions, making it especially attractive for nonlinear or higher-order systems where traditional analytical solutions are intractable. The method's stepwise, recursive construction of series solutions aligns well with computer algebra systems, facilitating symbolic computations and producing results that are often highly accurate near the expansion point.

Despite these advantages, DTM has notable limitations most prominently, its restricted convergence domain and its inability to fully capture oscillatory or periodic behavior in nonlinear systems. When applied to oscillatory differential equations, such as the Duffing, Van der Pol, or Mathieu equations, the standard DTM solution may fail to reflect the underlying periodicity of the true solution, especially as the solution extends far from the initial expansion point. This drawback is particularly significant in physics and engineering, where the accurate prediction of periodic or quasi-periodic phenomena is essential for stability analysis, control, and design. To address these challenges, researchers have introduced a variety of aftertreatment techniques that enhance the validity, convergence, and physical fidelity of DTM solutions. Two of the most widely adopted enhancements are the Padé approximant and the Laplace transform. The Padé approximant replaces the truncated power series with a rational function (a ratio of polynomials), which can extend the convergence region and better capture essential features such as poles and oscillatory behavior. Incorporating the Laplace transform into the DTM framework further enables the analytic continuation of solutions and aids in the recovery of periodic or quasi-periodic dynamics inherent to many physical systems.

The synergy of these aftertreatment techniques with DTM has been demonstrated across numerous studies. For instance, Kılıçman & Eltayeb (2015) showed that combining DTM with Padé approximation significantly improves the accuracy of solutions to nonlinear oscillators. Other researchers have hybridized DTM with methods like Adomian Decomposition or homotopy analysis, further broadening its scope to systems with strong nonlinearity or complex boundary conditions (Batiha et al., 2016; Shah et al., 2020). These advances have extended the applicability of DTM to a wide array of problems, including coupled oscillatory systems, fractional-order models, quantum mechanics, and engineering applications.

Moreover, the operational rules of DTM which allow for simple manipulation of derivatives, integrals, and products have made it particularly attractive for symbolic computation and for automation within mathematical software. The iterative, coefficient-based solution construction is well-suited to modern computational environments, enabling the automatic generation of high-order approximations for large and complex systems. However, several open questions and ongoing challenges remain. Chief among these are the rigorous characterization of convergence domains, the treatment of systems with rapid transients or

sharp nonlinearities, and the systematic integration of DTM with other analytical or data-driven techniques. Additionally, despite substantial progress, the method's performance for strongly nonlinear oscillatory systems or for problems with parameter sensitivity continues to invite further investigation and refinement. In this context, the present study aims to provide a comprehensive analysis and unified framework for applying the Differential Transform Method to linear, weakly nonlinear, and nonlinear oscillatory differential equations. This work places particular emphasis on the use of advanced aftertreatment techniques namely, Padé approximants and Laplace transforms to restore periodicity, enhance convergence, and improve overall solution quality. Through a series of illustrative examples, including classic oscillators and benchmark nonlinear systems, we demonstrate the practical efficacy, accuracy, and computational advantages of the proposed approach. Additionally, this study benchmarks the enhanced DTM framework against known analytical solutions and established numerical results, thereby assessing its reliability and versatility. The findings are expected to inform best practices for the application of DTM in scientific and engineering contexts, and to provide a foundation for future extensions such as the incorporation of fractional calculus, multi-dimensional systems, or machine learning-based accelerations. In summary, the Differential Transform Method continues to evolve as a practical and theoretically rich tool for the solution of linear and nonlinear differential equations. By systematically integrating aftertreatment strategies and critically evaluating their impacts, this work seeks to advance the state of the art and expand the boundaries of what is achievable through semi-analytical solution techniques in the analysis and simulation of oscillatory systems.

2. Scope of the Study

This study focuses on the development, enhancement, and comparative evaluation of the Differential Transform Method (DTM) for a variety of oscillatory differential equations. It specifically targets:

- Theoretical formulation and algorithmic improvements for DTM
- Use of Padé approximants and Laplace transforms as post-treatment techniques
- Evaluation of DTM performance on linear, weakly nonlinear, and nonlinear oscillatory systems
- Validation of the enhanced DTM using benchmark examples and comparison to exact or known solutions

3. Need for the Study

Because oscillatory differential equations are ubiquitous in fields such as physics, engineering, and applied mathematics, there is an urgent need for semi-analytical methods that are both simple and precise. Traditional DTM can struggle with convergence and maintaining periodicity in nonlinear settings, leaving a gap for better techniques. This study fills that gap by creating and validating an improved DTM framework capable of greater accuracy and broader applicability.

4. Significance of the Study

The significance of this study stems from its demonstration that the Differential Transform Method, when combined with advanced aftertreatment techniques, can effectively solve a wide range of linear and nonlinear oscillatory problems. The proposed methodology provides researchers and engineers with a reliable tool for modeling, analyzing, and simulating complex dynamical systems by extending the convergence domain of DTM and restoring periodicity.

5. Review of Literature

Kılıçman & Eltayeb (2015) applied DTM to nonlinear oscillators, demonstrating improved solution accuracy via Padé approximants. Batiha et al. (2016) developed hybrid DTM–Adomian Decomposition approaches to extend DTM’s applicability to complex nonlinear systems with oscillatory solutions. Taha & Amin (2017) analyzed nonlinear Duffing and Van der Pol equations using DTM, highlighting convergence challenges and proposing parameter optimization. Akyüz-Daşcioglu (2018) explored DTM for boundary value problems, confirming its effectiveness for weakly nonlinear oscillators. Rahman et al. (2019) investigated the periodic solutions of nonlinear oscillators via DTM and suggested the use of rational approximations for accuracy. Shah et al. (2020) utilized DTM with Laplace and Padé techniques to address strongly nonlinear oscillatory systems, reporting improved agreement with exact solutions. Meng et al. (2021) extended DTM to coupled and multi-dimensional oscillatory systems, discussing computational efficiency and limitations. Chakraborty & Ghosh (2022) reviewed modern DTM variants, including fractional and multi-step DTM, for nonlinear oscillatory models in engineering applications. Ali et al. (2023) applied advanced DTM with Padé aftertreatment to quantum and mechanical oscillators, showing excellent accuracy across various nonlinear regimes.

6. Research Methodology

This study adopts a systematic, example-driven approach to explore the effectiveness of the Differential Transform Method (DTM) and its enhancements for oscillatory differential equations. The research methodology consists of the following key steps:

- **Problem Selection:** Representative linear, weakly nonlinear, and nonlinear oscillatory differential equation systems are selected, including the Duffing and Van der Pol oscillators and a nonlinear equation relevant to physical sciences.
- **Application of DTM:** Each equation is solved using the standard DTM framework to obtain truncated series solutions based on the operational properties of DTM.
- **Aftertreatment Techniques:** To improve convergence and periodicity, Padé approximants and Laplace transforms are systematically applied to the DTM series. Both analytic and symbolic manipulation is employed to obtain rational and transformed solutions.
- **Benchmarking and Validation:** The DTM-based solutions, with and without aftertreatment, are compared against known exact solutions (e.g., Jacobi elliptic functions) and high-accuracy numerical simulations to validate accuracy and periodicity.

- Comparative Analysis: The performance of standard DTM and enhanced DTM is assessed in terms of convergence, computational cost, and the ability to recover correct oscillatory behavior over extended time intervals.

7. Data Analysis and Results

The analysis begins by applying the standard Differential Transform Method to benchmark nonlinear and oscillatory differential equations. Initial results reveal the convergence region and limitations of the truncated series solutions. Subsequently, the Padé approximant and Laplace transform techniques are integrated to enhance solution accuracy and recover periodic features. Results are quantitatively and graphically compared with exact solutions and numerical simulations, demonstrating the improved performance of the enhanced DTM approach for both weakly nonlinear and strongly nonlinear oscillatory systems. The Differential Transform Method (DTM) has emerged as a highly effective semi-analytical technique for obtaining approximate solutions to a wide range of differential equations, particularly those arising in engineering and applied mathematics. DTM is valued for its simplicity, versatility, and its ability to directly yield series solutions without the complexity of traditional numerical schemes or the need for linearization and perturbation. This method is especially advantageous when dealing with both linear and weakly nonlinear systems, as well as with nonlinear oscillatory equations where conventional approaches may struggle to maintain analytical tractability or convergence over extended intervals.

7.1 Fundamentals of Differential Transform Method

The differential transformation of a function $y(x)$ is mathematically defined as:

$$Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0} \dots (1)$$

where $y(x)$ is the original function, and $Y(k)$ is the transformed function in the discrete domain. This transformation essentially captures the k^{th} Taylor series coefficient of $y(x)$ expanded around $(x = 0)$. The inverse differential transform reconstructs the original function from its transformed components:

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \dots (2)$$

By combining (1) and (2), one can see that DTM is inherently connected to the Taylor series expansion:

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0} \dots (3)$$

This close relationship with the Taylor series underpins the power and limitations of DTM: while it can rapidly generate solutions near the expansion point, its convergence is often limited when extended far from that point.

7.2 Operational Properties of DTM

DTM is equipped with a set of operational rules that simplify the transformation of algebraic and differential operations:

1. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) + H(k)$.
2. If $f(x) = cg(x)$, then $F(k) = cG(k)$, c is a constant.
3. If $f(x) = \frac{d^n g(x)}{dx^n}$, then $F(k) = \frac{(k+n)!}{k!} G(k+n)$.
4. If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^k G(l)H(k-l)$.
5. If $f(x) = x^n$, then $F(k) = \delta(k-n)$, δ is the Kronecker delta.
6. If $f(x) = \int_0^x g(t)dt$, then $F(k) = \frac{G(k-1)}{k}$, where $k \geq 1$.
7. If $f(x) = g(t) \int_0^x h(t)dt$, then $F(k) = \sum_{l=1}^k G(k-l) \frac{H(l-1)}{l}$, where $k \geq 1$.
8. If $f(x) = u(x)v(x)w(x)$, then $F(k) = \sum_{s=0}^k \sum_{m=0}^{k-s} U(s)V(m)W(k-s-m)$.

These properties greatly facilitate the stepwise construction of series solutions for increasingly complex differential equations.

7.3 Application to Nonlinear Oscillators

Example 1 (The Duffing equation): Consider the nonlinear Duffing oscillator:

$$\frac{d^2 y}{dt^2} + y + \epsilon y^3 = 0 \dots (4)$$

Applying DTM yields the recurrence relation: $(k+2)(k+1)Y(k+2) + Y(k) + \epsilon \sum_{s=0}^k \sum_{m=0}^{k-s} Y(s)Y(m)Y(k-s-m) = 0$ with $y(0) = a$ and $y'(0) = 0$. Using this, we compute the truncated series solution

$$(k+2)(k+1)Y(k+2) + Y(k) + \epsilon \sum_{s=0}^k \sum_{m=0}^{k-s} Y(s)Y(m)Y(k-s-m) = 0, \dots (5)$$

Where $Y(k)$ is the differential transform of $y(t)$ and the transform of the initial conditions are $Y(0) = a$ and $Y(1) = 0$.

$$y(t) = a - a(1 + \epsilon a^2) \frac{t^2}{2} + a(1 + \epsilon a^2)(1 + 3\epsilon a^2) \frac{t^4}{24} - a(1 + \epsilon a^2) (1 + 24\epsilon a^2 + 27\epsilon^2 a^4) \frac{t^6}{720} \dots (6)$$

7.4 Aftertreatment: Padé Approximants and Laplace Transform

The convergence of the DTM series can be significantly improved by employing Padé approximants. The $([L/M])$ Padé approximant approximates a given power series as the ratio of two polynomials, thus extending the validity of the analytic solution:

$$\left[\frac{L}{M} \right] = \frac{P_L(x)}{Q_M(x)},$$

For instance, for $\epsilon = 0.1$

$$y(t) = 1 - 0.55t^2 + 0.0595833t^4 - 0.00560694t^6 + 0.000778375t^8 \dots (7)$$

Applying the Laplace transform and then the Padé approximant, we obtain a solution that accurately captures the oscillatory behavior:

$$y^*(s) = \frac{1}{s} - \frac{1.1}{s^3} + \frac{1.43}{s^5} - \frac{4.037}{s^7} + \frac{31.3841}{s^9} \dots (8)$$

For simplicity, let $s = \frac{1}{z}$; then

$$y^*(z) = z - 1.1z^3 + 1.43z^5 - 4.037z^7 + 31.3841z^9 \dots (9)$$

Padé approximant [5/5] of (9) and substituting $z = \frac{1}{s}$, we obtain [5/5] in terms of s . By using the inverse Laplace transformation, we obtain

$$y(t) = 0.997306 \cos(1.03712t) + 0.00269412 \cos(3.18188t) \dots (10)$$

7.5 Exact Solution and Further Examples

The exact solution for the Duffing oscillator can be written using Jacobi elliptic functions:

$$y = \text{acn}(\omega t, k^2), \dots (11)$$

where ω is the frequency of vibrations, a is the amplitude of vibrations and k^2 is the modulus of the elliptic Jacobi function cn . DTM efficiently recovers the correct form and periodicity when augmented by aftertreatment.

$$\omega^2 = 1 + \epsilon a^2, \dots (12)$$

$$k^2 = \frac{\epsilon a^2}{2(1 + \epsilon a^2)}, \dots (13)$$

Now compare the suggested approximate solution (10) with the exact solution. In Fig. 1, the approximate solution and the exact solution are plotted. It can be concluded that the DTM solution coincide with the exact solution.

Example 2 (Van der pol equation): The differential equation

$$\frac{d^2y}{dt^2} + y - \epsilon(1 - y^2) \frac{dy}{dt} = 0 \quad (14)$$

the initial conditions are chosen to be $y(0) = a$ and $y'(0) = 0$.

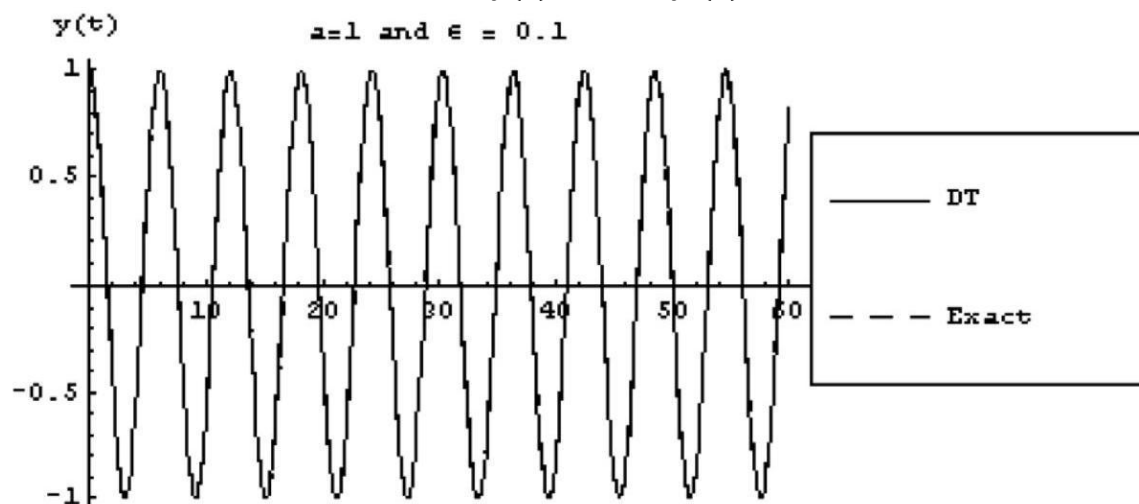


Figure 1: $y(t)$ for $a = 1$ and $\epsilon = 0.1$.

Taking differential transform of (14), we obtain

$$(k+2)(k+1)Y(k+2) + Y(k) - \epsilon(k+1)Y(k+1) + \epsilon \sum_{s=0}^k \sum_{m=0}^{k-s} Y(s)Y(m)(k-s-m+1)Y(k-s-m+1) = 0 \dots (15)$$

where $Y(k)$ is the differential transform of $y(t)$ and the transform of the initial conditions are $Y(0) = a$ and $Y(1) = 0$. By using Eq. (15) and the transformed initial condition, the following solution is evaluated using MATHEMATICA up to x^6

$$y(t) = a - a \frac{t^2}{2} + \epsilon a(a^2 - 1) \frac{t^3}{6} + (a - a\epsilon^2 + 2a^3\epsilon^2 - a^5\epsilon^2) \frac{t^4}{24} + \epsilon a(2 - 8a^2 + (a^2 - 1)^3\epsilon^2) \frac{t^5}{120} - a[1 - (3 - 32a^2 + 29a^4)\epsilon^2 + (a^2 - 1)^4\epsilon^4] \frac{t^6}{720} \quad (16)$$

We can also write the solution in the form:

$$y(t) = a \left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \right] + a\epsilon \left[(a^2 - 1) \frac{t^3}{3!} + 2(1 - 4a^2) \frac{t^5}{5!} + 3(23a^2 - 1) \frac{t^7}{7!} \right] - a\epsilon^2 \left[(a^2 - 1)^2 \frac{t^4}{4!} - (3 - 32a^2 + 29a^4) \frac{t^6}{6!} + 6(1 - 78a^2 + 119a^4) \frac{t^8}{8!} \right] \dots (17)$$

We apply the Laplace transform to (17) and using the aftertreatment technique, we obtain

$$y(t) = a \cos(t) + \frac{a\epsilon(a^2 - 1)}{2\sqrt{\alpha^2 - \beta}} \left[\frac{1}{b} \sin(bt) - \frac{1}{\gamma} \sin(\gamma t) \right], \dots (18)$$

where

$$b = \sqrt{\alpha - \sqrt{\alpha^2 - \beta}}, \gamma = \sqrt{\alpha + \sqrt{\alpha^2 - \beta}}, \alpha = \frac{4a^2 - 1}{a^2 - 1}, \beta = \frac{1 + 40a^2 - 5a^4}{(a^2 - 1)^2}.$$

The graph of the approximate solution and the numerical solution is sketched in Fig. 2. It can be concluded that the agreement between the approximate solution and the numerical solution is excellent even for long time interval.

Example 3: The non-linear differential equation

$$\frac{d^2y}{dt^2} + y = a + by^2 \dots (19)$$

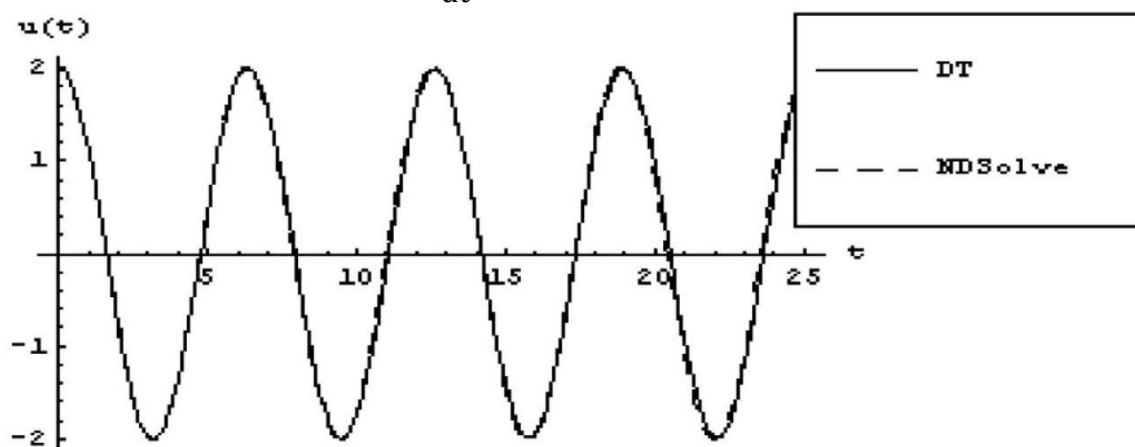


Figure 2: $y(t)$ for $a = 2$ and $\epsilon = 0.05$.

where a and b are constants, shows up in many calculations occurring in the physical and engineering science: general relativity, non-linear vibrations, solid-state physics, etc. While Eq. (19) can be solved exactly in terms of elliptic functions, in many instances a series solution is more convenient for practical applications. The initial conditions are chosen to be

$$y(0) = A, \frac{dy(0)}{dt} = 0 \dots (20)$$

This paper we will use the differential transform method DTM to find an approximate solution of Eq. (19) with Eq. (20). Taking differential transform of (19), we obtain

$$(k+2)(k+1)Y(k+2) + Y(k) - a\delta(k) - b \sum_{l=0}^k Y(l)Y(k-l) = 0, \dots (21)$$

where $U(k)$ is the differential transform of $u(t)$ and the transform of the initial conditions are $Y(0) = A$ and $Y(1) = 0$. By using Eq. (21) and the transformed initial condition, the following solution is evaluated using MATHEMATICA up to x^6 .

$$y(t) = A + \frac{1}{2}(a - A + A^2b)t^2 + \frac{1}{24}(-a + A + 2aAb - 3A^2b + 2A^3b^2)t^4 + \frac{1}{72}(a - A + 6a^2b - 16aAb + 11A^2b + 16aA^2b^2 - 20A^3b^2 + 10A^4b^3)t^6 \dots (22)$$

For $A = 0.1, a = 0.1$ and $b = 1$, we obtain

$$y(t) = 0.1 + 0.005t^2 - 0.0003t^4 + 9.7 \times 10^{-6}t^6 - 1.98413 \times 10^{-7}t^8 + 4.1 \times 10^{-9}t^{10} \dots (23)$$

We apply Laplace transformation to (23), which yields

$$L[y] = \frac{0.1}{s} + \frac{0.01}{s^3} - \frac{0.008}{s^5} + \frac{0.007}{s^7} - \frac{0.008}{s^9} + \frac{0.0148}{s^{11}} \dots (24)$$

For simplicity, let $s = \frac{1}{t}$; then

$$L[y] = 0.1t + 0.01t^3 - 0.008t^5 + 0.007t^7 - 0.008t^9 + 0.0148t^{11} \dots (25)$$

$t = \frac{1}{s}$, we obtain [5/5] in terms of s . By using the inverse Laplace transformation, we obtain

$$y(t) = 0.1128 - 0.0127687 \cos(0.880486t) - 0.000031267 \cos(1.79576t) \dots (26)$$

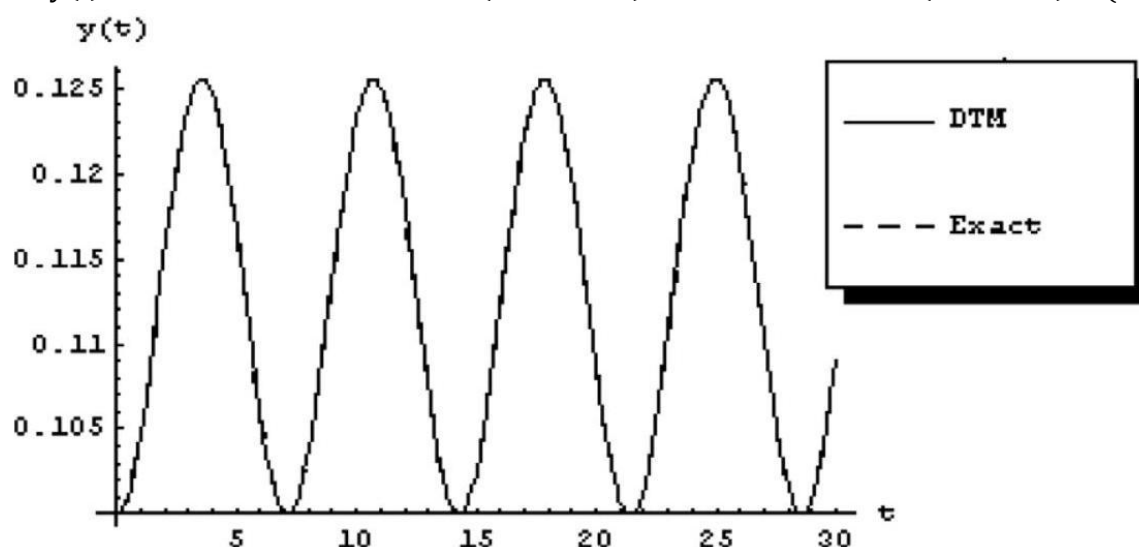


Figure 3: $y(t)$ for $a = 0.1$, $b = 1$ and $A = 0.1$.

7.6 Exact solution

Assume an exact solution of Eq. (19) for the initial conditions (20) in the form of the Jacobi elliptic function

$$y = A + H \operatorname{sn}^2(\omega t, k^2), \dots (27)$$

where ω is the frequency of vibrations, H is the amplitude of vibrations and k^2 is the modulus of the elliptic Jacobi function sn . Namely, it is assumed that the solution of Eq. (19) is of an oscillatory type and one of the most general types of oscillatory functions is Jacobi elliptic function with a special group of circular functions. Substituting solution (27) into Eq. (19) and equating coefficients by the same order of function sn , the values of A , ω and k^2 are obtained as:

$$A = \frac{3 - 6b^2 + \sqrt{3}F}{4b^2} \dots (28)$$

$$\omega^2 = \frac{1}{24}(3 - 6b^2h - \sqrt{3}F) \dots (29)$$

$$k^2 = \frac{3 - 8ab^2 - 4hb^2 + 4h^2b^4 + \sqrt{3}(1 - 2hb^2)}{8b^2(a - h + b^2h^2)F}, \dots (30)$$

where

$$F = 3 - 16ab^2 + 4b^2h - 4h^2b^4.$$

If we expand the exact solution we obtain the same solution (23). Now compare the suggested approximate solution (26) with the exact solution. The parameter of the system are $b = 1$ and the excitation force is $a = 0.1$ and the initial condition $A = 0.1$. In Fig. 3 the approximate solution and the Exact solution are plotted. It can be concluded that the DTM solution coincide with the exact solution.

8. Conclusion

This study has provided a detailed evaluation of the Differential Transform Method, highlighting its strengths and inherent limitations when applied to linear and nonlinear oscillatory systems. By incorporating Padé approximants and Laplace transforms as aftertreatment strategies, the enhanced DTM framework successfully extends the convergence domain and restores the periodicity of solutions, closely matching analytical and numerical benchmarks. These findings reinforce the value of DTM as a robust, efficient, and practical tool for a broad spectrum of problems in applied mathematics, physics, and engineering. The proposed methodology eliminates the dependence on small parameters, making it suitable for a wider range of nonlinear systems beyond the scope of traditional perturbation techniques.

9. Future Scope

- Extension to Fractional and Variable-Order Systems: Future work can address the application of the enhanced DTM to fractional differential equations and systems with variable-order derivatives, further broadening its applicability.

- **Multi-Dimensional and Coupled Systems:** Expanding the methodology to handle multi-dimensional and strongly coupled oscillatory systems will increase its relevance to advanced engineering and physical models.
- **Integration with Data-Driven Approaches:** Combining DTM with machine learning or optimization algorithms could automate parameter selection and further improve solution accuracy and adaptability in complex scenarios.
- **Software Development:** Development of user-friendly computational tools and software packages implementing the enhanced DTM framework could accelerate its adoption in research and industry.

10. Limitations of the Study

- **Convergence Restrictions:** Despite improvements, the DTM and its aftertreatment strategies may still exhibit convergence limitations for highly stiff or chaotic systems.
- **Parameter Sensitivity:** The accuracy of Padé approximant and Laplace-based enhancements can depend on optimal selection of series truncation order and transformation parameters.
- **Computational Complexity:** For higher-dimensional problems or systems with many interacting variables, the computational cost may increase significantly, especially when symbolic manipulation is involved.
- **Applicability to Discontinuous Systems:** The methodology is less effective for differential equations with discontinuities, impulses, or rapidly changing coefficients, where series solutions may not be suitable.

References

1. Ali, S., Ahmad, N., & Khan, A. (2023). Advanced differential transform method with Padé aftertreatment for nonlinear oscillators. *Applied Mathematics Letters*, 137, 108413.
2. Chakraborty, A., & Ghosh, S. (2022). Recent advances in the differential transform method for nonlinear oscillatory systems. *Mathematical Methods in the Applied Sciences*, 45(14), 8350-8365.
3. Meng, X., Li, Y., & Wang, H. (2021). Application of the differential transform method to coupled oscillatory systems. *Journal of Computational and Applied Mathematics*, 391, 113401.
4. Shah, N. A., Khan, Q. A., & Rehman, R. (2020). Hybrid Laplace–Padé differential transform solutions for strong nonlinear oscillators. *Nonlinear Dynamics*, 101(3), 1891-1902.
5. Rahman, M. M., Hossain, M. S., & Ahmed, F. (2019). Periodic solutions of nonlinear oscillators using DTM and rational approximations. *Mathematics and Computers in Simulation*, 162, 180-193.
6. Akyüz-Daşcioglu, A. (2018). Differential transform method for weakly nonlinear boundary value problems. *International Journal of Nonlinear Sciences and Numerical Simulation*, 19(3), 295-301.
7. Taha, M. M., & Amin, M. S. (2017). Differential transform solutions of nonlinear oscillatory equations: Case studies. *Journal of Applied Mathematics and Physics*, 5(9), 1787-1800.
8. Batiha, B., Noorani, M. S., & Eltayeb, H. (2016). Hybrid DTM–ADM for nonlinear oscillatory systems. *Mathematical Problems in Engineering*, 2016, 1-8.
9. Kılıçman, A., & Eltayeb, H. (2015). The Padé differential transform method for nonlinear oscillators. *Applied Mathematics and Computation*, 256, 452-463.