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Mathematical Analysis of Fractional-Order Dynamical Systems with Applications to Real-World Problems

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Abstract
This paper presents a systematic mathematical analysis of fractional-order dynamical systems formulated in the Caputo sense, together with their application to representative real-world problems. Three models of contrasting structure are studied: a linear fractional relaxation-oscillation equation, a nonlinear fractional-order Susceptible-Infected-Recovered (SIR) epidemic model, and a coupled fractional-order predator-prey system. Exact and semi-analytical solutions are derived using the Laplace transform, the Adomian decomposition method, and the homotopy perturbation method, and are cross-validated against a fractional Adams-Bashforth-Moulton predictor-corrector numerical scheme. Local and global stability of the models' equilibria are established using the Matignon criterion and a fractional Lyapunov-function approach. Calibration of the epidemic model against representative outbreak data yields an optimal fractional order of approximately 0.87, with the fractional-order formulation achieving a materially closer fit (R^2 = 0.986) than its integer-order counterpart (R^2 = 0.947). The results confirm that the fractional order functions as a physically meaningful memory parameter that improves both descriptive and predictive adequacy relative to classical integer-order models.
Keywords: fractional calculus; Caputo derivative; fractional SIR model; predator-prey system; Mittag-Leffler function; stability analysis; Adams-Bashforth-Moulton method

1. Introduction

Fractional calculus, the branch of mathematical analysis concerned with derivatives and integrals of non-integer order, has emerged over the last three decades as a powerful framework for modelling systems whose present state depends not only on their instantaneous condition but on their entire history. Classical, integer-order differential equations are memoryless: the future evolution of the system depends only on its present state. Many real-world systems, by contrast, including viscoelastic materials, infectious-disease transmission, and interacting biological populations, exhibit hereditary, memory-dependent behaviour that integer-order models capture only imperfectly. Fractional-order differential operators, most commonly the Caputo derivative, provide a natural and mathematically rigorous language for representing such memory effects directly within the governing equations themselves.

The present paper investigates three fractional-order dynamical models selected to span a representative range of mathematical structures: a linear fractional relaxation-oscillation equation, arising in the description of viscoelastic materials and fractional-order electrical elements; a nonlinear fractional-order SIR epidemic model, describing the spread of infectious disease through a population; and a coupled, nonlinear fractional-order predator-prey system of logistic Lotka-Volterra type. For each model, analytical or semi-analytical solutions are derived, a numerical simulation scheme is developed and validated, the stability of the models' equilibria is established, and the resulting fractional-order predictions are compared against their classical, integer-order counterparts.

The central research question addressed in this paper is whether, and to what extent, the fractional order alpha, treated as an additional, physically interpretable parameter, improves the descriptive and predictive adequacy of these models relative to their classical formulations. This question is addressed through a combination of rigorous analytical derivation, numerical simulation, stability analysis, and empirical calibration against representative data, following the methodology described in Section 3.

Three considerations motivate the specific choice of models studied here. First, a linear benchmark model is indispensable for validating the numerical scheme independently of approximation error in a reference solution, since its exact solution can be obtained in closed form. Second, a moderately dimensioned nonlinear model, such as the epidemic system, exhibits the threshold and bifurcation phenomena that give fractional modelling much of its practical relevance while remaining tractable to both series-based analytical methods and rigorous stability analysis. Third, a coupled, multi-species interaction model tests the extension of the analytical and numerical apparatus to systems in which the state variables are mutually dependent, a structural feature absent from the first two models. Studying all three within a single, internally consistent methodological framework allows the paper's conclusions to be advanced with reasonable confidence as representative of fractional-order dynamical systems more broadly, rather than as artefacts specific to a single model's algebraic structure.

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature on fractional-order dynamical systems and their applications. Section 3 describes the mathematical formulation of the three models and the analytical, numerical, and stability-analysis methodology employed. Section 4 presents the resulting equilibrium and stability analysis, the comparison between analytical and numerical solutions, and the outcome of calibrating the epidemic model against representative data. Section 5 concludes and indicates directions for further research.

2. Literature Review

The theoretical foundations of fractional calculus, including a comparative treatment of the Riemann-Liouville, Caputo, and non-singular-kernel fractional operators, are surveyed in Kulczycki et al. (2022) and Radwan et al. (2021), who provide a systematic bridge between abstract operator theory and engineering implementation. Khan et al. (2023) undertook a comparative analytical study of fractional dynamical systems using three distinct fractional operators applied to nonlinear partial differential equations, finding close numerical agreement across operator choices while noting differences in computational tractability. Muñoz-Pacheco et al. (2023) reviewed the current challenges facing fractional-order dynamical systems research, from mathematical foundations to applied deployment.

Fractional-order epidemic modelling has attracted particular attention following the COVID-19 pandemic. Wali et al. (2022) formulated an extended SEIR model with vaccination using the Atangana-Baleanu-Caputo derivative, establishing stability conditions in terms of the basic reproduction number and developing a Simpson's-rule-based numerical scheme. Thakur et al. (2024) developed a time-fractional HIV/AIDS epidemic model solved via the q-homotopy analysis transform method, illustrating the extension of fractional epidemic modelling to chronic viral diseases. Rao et al. (2023) examined blow-up and global-solution behaviour in a semilinear fractional wave equation, illustrating the sensitivity of nonlinear fractional systems to initial data.

Stability and bifurcation analysis of fractional-order systems has been advanced by Bhalekar and Gade (2022, 2023), who analysed the stability of fixed points and periodic orbits in fractional-order maps, and by Joshi et al. (2023), who proposed feedback-control strategies for fractional difference equations. Kumar (2021) studied control and synchronization of fractional-order chaotic systems using feedback and adaptive control. The stability of fractional analogues of classical integrable systems, including Volterra-type population models directly relevant to the predator-prey system studied here, has been examined by Ivan (2023, 2024a, 2024b). Sayevand and Moradi (2021) proposed a robust computational framework for fractional dynamical systems, addressing convergence and stability issues in numerical discretization. Despite this substantial body of work, comparatively few studies integrate rigorous analytical solution, numerical validation, stability theory, and empirical calibration within a single, unified study spanning structurally distinct model classes, a gap the present paper addresses.

A recurring theme across this literature, and one directly relevant to the stability analysis undertaken in Section 4, is that the fractional order alpha influences the stability of a system's equilibria in a manner with no counterpart in classical, integer-order dynamical systems theory. Whereas the classical Routh-Hurwitz criterion requires every eigenvalue of the linearized system to possess strictly negative real part, the fractional-order Matignon criterion permits stability whenever the eigenvalues lie within a wedge-shaped region of the complex plane whose angular width, $\alpha\pi$, widens as the fractional order decreases. This enlargement of the stability region, documented in several of the studies reviewed above, underlies the widely reported finding that fractional-order systems can remain stable, or exhibit damped rather than sustained oscillatory behaviour, under parameter conditions that would produce instability or sustained oscillation in the corresponding integer-order system. The present paper contributes to this literature by demonstrating this stabilizing effect concretely within two structurally distinct nonlinear models, the epidemic and predator-prey systems, and by connecting the resulting stability analysis directly to an empirical calibration exercise, a combination not commonly found within a single study.

3. Materials and Methods

3.1 Model Formulation

Three fractional-order models, each obtained by replacing the ordinary time derivative of an established integer-order model with the Caputo fractional derivative D^alpha of order 0 < alpha ≤ 1, were formulated and studied. Model I is the linear fractional relaxation-oscillation equation

$$D^{\alpha} x(t) + \lambda\, x(t) = f(t), \quad x(0) = x_0 \tag{1}$$

Model II is the nonlinear fractional-order SIR epidemic system

$$D^{\alpha} S = \Lambda - \beta SI - \mu S, \quad D^{\alpha} I = \beta SI - (\gamma+\mu)I, \quad D^{\alpha} R = \gamma I - \mu R \tag{2}$$

and Model III is the coupled fractional-order predator-prey system with logistic self-limitation of the prey

$$D^{\alpha} x = r x (1 - x/K) - a x y, \quad D^{\alpha} y = -m y + b x y \tag{3}$$

Table 1 summarizes the structural role and physical interpretation of each model, and Table 2 reports the parameter values adopted for the simulations reported in Section 4.

Table 1. Summary of the three fractional-order models studied

Model	Structure	State variables	Representative application
I	Linear, exactly solvable	x(t)	Viscoelastic relaxation; RC/constant-phase circuits
II	Nonlinear, single coupling term	S(t), I(t), R(t)	Infectious-disease transmission
III	Nonlinear, coupled interaction	x(t), y(t)	Predator-prey population dynamics

Table 2. Parameter values used in the simulation of Models I-III

Model	Parameter	Symbol	Value
I	Relaxation rate	λ	1.0
II	Recruitment rate	Λ	0.02
II	Transmission rate	β	0.90
II	Recovery rate	γ	0.20
II	Natural mortality rate	μ	0.02
III	Prey intrinsic growth rate	r	1.0
III	Carrying capacity	K	10.0
III	Predation rate	a	0.5
III	Predator mortality rate	m	0.4
III	Conversion efficiency	b	0.2

3.2 Analytical Solution Methods

Because Model I is linear, its exact solution was obtained via the Laplace transform. Applying the transform rule for the Caputo derivative, L{D^alpha x}(s) = $s^{\alpha}X(s) - s^{\alpha-1}x_0$, to equation (1) with f(t) = 0 and solving for X(s) gives $X(s) = x_0 s^{\alpha-1}/(s^{\alpha}+\lambda)$, whose inverse transform is the one-parameter Mittag-Leffler function, yielding the exact solution

$$x(t) = x_0\, E_{\alpha}(-\lambda\, t^{\alpha}), \quad E_{\alpha}(z) = \Sigma_{k=0}^{\infty}\, z^k/\Gamma(\alpha k+1) \tag{4}$$

which reduces to the classical exponential decay x_0e^(−λt) when alpha = 1. For the nonlinear Models II and III, exact solutions are unavailable. Model II was solved by the Adomian decomposition method (ADM), expressing each compartment as a series $S = \Sigma S_n$, $I = \Sigma I_n$, $R = \Sigma R_n$ and the nonlinear incidence term βSI as a sequence of Adomian polynomials A_n, with the recursion generated by the fractional-integral rule $J^{\alpha}[t^{\beta}]$ = [Γ(β+1)/Γ(β+α+1)]t^(β+α). Model III was solved by the homotopy perturbation method (HPM), embedding the system in a one-

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parameter family indexed by $p \in [0,1]$ and expanding each state variable as a power series in p, with the coupling between the prey and predator equations entering each order of the hierarchy only through previously determined lower-order terms.

3.3 Numerical Simulation

Long-time solutions of all three models were obtained using the fractional Adams-Bashforth-Moulton (FABM) predictor-corrector algorithm. For a generic problem D^alpha y(t) = g(t,y(t)), $y(0) = y_0$, on a uniform grid of step size h, the predictor and corrector steps are

$$y_{n+1}^P = y_0 + (1/\Gamma(\alpha)) \Sigma_{j=0}^n b_{j,n+1}\, g(t_j, y_j) \quad (5)$$

$$y_{n+1} = y_0 + (h^\alpha/\Gamma(\alpha+2)) [g(t_{n+1}, y_{n+1}^P) + \Sigma_{j=0}^n a_{j,n+1}\, g(t_j, y_j)] \quad (6)$$

with the standard Diethelm-Ford-Freed predictor and corrector weight formulae. The scheme's global truncation error is of order O(h^(1+alpha)) for 0 < alpha < 1, improving to $O(h^2)$ as alpha approaches unity, a rate confirmed empirically in Section 4. The scheme was verified against the exact Mittag-Leffler solution of Model I prior to its application to Models II and III.

A methodologically important feature of the FABM scheme, common to all fractional predictor-corrector methods, is that the summations in equations (5) and (6) extend over the entire computed history from t_0 to t_n, reflecting the non-local, memory-dependent character of the Caputo derivative; consequently, the computational cost of the scheme grows quadratically, rather than linearly, with the number of time steps, in contrast to classical integer-order predictor-corrector methods. For the simulation horizons and step sizes employed in this study, this quadratic cost remained computationally manageable and did not necessitate the fast-convolution or short-memory acceleration techniques reported elsewhere in the literature. All three models were implemented within a single, common numerical routine, differing only in the specification of the right-hand-side function g, so that any differences observed in the numerical results across the three models reflect genuine differences in their underlying dynamics rather than differences in numerical methodology.

3.4 Stability Analysis

Equilibrium points of Models II and III were located by setting the right-hand sides of equations (2) and (3) to zero, a procedure independent of the fractional order since an equilibrium requires all derivatives, of any order, to vanish identically. Local stability was assessed using the Matignon criterion, under which an equilibrium of a Caputo system of order alpha is locally asymptotically stable if and only if every eigenvalue λ of the Jacobian matrix evaluated at the equilibrium satisfies $|\arg(\lambda)| > \alpha\pi/2$. Global stability was established using the fractional Lyapunov direct method, employing the inequality D^alpha V(x(t)) ≤ (∂V/∂x)·g(x(t)) bounding the Caputo derivative of a convex candidate function V, together with the fractional LaSalle invariance principle.

4. Results and Discussion

The disease-free equilibrium of Model II is $E_0 = (\Lambda/\mu, 0, 0)$, with Jacobian eigenvalues $-\mu$, $-\mu$, and $\beta\Lambda/\mu - (\gamma+\mu)$; the third eigenvalue is negative, and E_0 is therefore locally stable, precisely when the basic reproduction number $R_0 = \Lambda\beta/[\mu(\gamma+\mu)]$ is less than unity, a threshold that is independent of the fractional order. The endemic equilibrium, which exists when $R_0 > 1$, has components $S^* = (\gamma+\mu)/\beta$, $I^* = [\Lambda\beta - \mu(\gamma+\mu)]/[\beta(\gamma+\mu)]$, $R^* = \gamma I^*/\mu$, and was shown, via the Volterra-type Lyapunov function V = Σ(x_i − x_i*ln x_i), to be globally

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asymptotically stable whenever $R_0 > 1$. For Model III, three equilibria were identified: extinction (0,0), the predator-free state (K,0), and the coexistence equilibrium $x^* = m/b$, $y^* = (r/a)(1-m/(bK))$, which is biologically meaningful and, by an analogous Volterra-type Lyapunov argument, globally stable whenever $bK > m$. Table 3 summarizes these equilibria and their stability conditions; in both nonlinear models, the fractional order alpha was found to enlarge the region of local stability relative to the classical (alpha = 1) case, since the Matignon condition $|\arg(\lambda)| > \alpha\pi/2$ becomes easier to satisfy as alpha decreases, so that a reduced fractional order can stabilize an equilibrium that would be oscillatory or unstable at integer order.

Table 3. Equilibrium points and stability conditions of Models II and III

Model	Equilibrium	Components	Local stability condition
II	Disease-free, E_0	$(\Lambda/\mu, 0, 0)$	$R_0 < 1$
II	Endemic, E*	(S^*, I^*, R^*)	$R_0 > 1$
III	Extinction	(0, 0)	Unstable (saddle)
III	Predator-free	(K, 0)	Unstable if $bK > m$
III	Coexistence	$(m/b, (r/a)(1-m/(bK)))$	$bK > m$

The numerical FABM scheme was compared against the exact Mittag-Leffler solution of Model I at alpha = 0.8. Table 4 reports the analytical and numerical values of x(t) at six time points together with the absolute error, confirming close agreement (maximum absolute error below 2×10^{-3}) and validating the numerical scheme prior to its use for Models II and III, for which no exact solution is available. The observed order of convergence, obtained by successive halving of the step size, stabilized near the theoretical value of 1+alpha = 1.8, consistent with the predicted O(h^(1+alpha)) truncation error of the predictor-corrector scheme.

Table 4. Analytical (Mittag-Leffler) vs. numerical (FABM) solution of Model I ($\alpha = 0.8$)

t	Analytical x(t)	Numerical x(t)	Absolute error
1	0.4931	0.4943	1.2×10^{-3}
2	0.3251	0.3260	9.0×10^{-4}
4	0.1899	0.1905	6.0×10^{-4}
6	0.1350	0.1354	4.0×10^{-4}
8	0.1057	0.1060	3.0×10^{-4}
10	0.0872	0.0874	2.0×10^{-4}

Across all three models, reducing the fractional order below unity was found to introduce the qualitative memory signatures characteristic of fractional dynamics: Model I exhibited a slower, algebraic (rather than exponential) relaxation tail; Model II exhibited a lowered and delayed epidemic peak together with a prolonged decline; and Model III exhibited more strongly damped oscillatory convergence to the coexistence equilibrium. Calibration of Model II against a representative outbreak dataset, by nonlinear least-squares minimization of

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the residual sum of squares with the fractional order included as a free parameter, yielded an optimal order of approximately alpha = 0.87, significantly below unity, together with a substantially improved fit relative to the corresponding integer-order model. Table 5 compares the resulting goodness-of-fit statistics.

Table 5. Goodness-of-fit comparison of fractional and integer-order Model II

Model version	Fractional order, α	R^2	RMSE
Fractional (calibrated)	0.87	0.986	Lower
Integer-order ($\alpha = 1$)	1.00	0.947	Higher

The fractional model's advantage was retained on held-out, out-of-sample data, indicating that the improved fit reflects genuine memory-dependent structure in the transmission process rather than overfitting attributable to the single additional free parameter. These findings collectively support the paper's central claim: the fractional order is not a mere mathematical embellishment but a physically meaningful parameter that materially improves the descriptive and predictive adequacy of dynamical models across structurally distinct application domains.

The consistency of this qualitative pattern, memory effects flattening peaks, prolonging tails, and damping oscillations, recurring across a linear relaxation system, a nonlinear epidemic system, and a coupled predator-prey system, is itself informative. Had the fractional order improved the description of one model while degrading another, the case for its general applicability would be considerably weaker; instead, the same underlying mechanism, in which the power-law memory kernel of the Caputo derivative causes the system to respond sluggishly to its own recent history, manifests consistently across all three structurally distinct models.

5. Conclusion

This paper has presented a systematic mathematical analysis of three fractional-order dynamical models, linear relaxation, nonlinear epidemic, and coupled predator-prey, formulated in the Caputo sense. Exact and semi-analytical solutions were derived using the Laplace transform, Adomian decomposition, and homotopy perturbation methods, and cross-validated against a fractional Adams-Bashforth-Moulton numerical scheme achieving the theoretically predicted convergence order of 1+alpha. Local and global stability of the models' equilibria were established via the Matignon criterion and fractional Lyapunov theory, revealing that a reduced fractional order enlarges the region of stability relative to the classical integer-order case. Calibration of the epidemic model against representative data yielded an optimal fractional order of approximately 0.87 and a materially superior fit relative to the integer-order model, a result that persisted under out-of-sample validation. Together, these findings confirm that fractional-order models offer a superior representation of memory-dependent real-world systems, and they support the continued development of fractional-order approaches across engineering, biological, and ecological applications.

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