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Performance Assessment of Iterative Numerical Techniques for Nonlinear Equation Solving and Optimization Problems

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Abstract

Nonlinear equations and optimization problems are central to engineering and scientific computing, yet they rarely admit closed-form solutions and must be solved by iterative numerical methods whose performance varies widely with problem structure. This study presents a systematic, controlled comparative assessment of seven prominent numerical techniques spanning four algorithmic families: bracketing methods (Bisection and Regula-Falsi), open methods (Newton-Raphson and Secant), fixed-point iteration with Aitken Δ^2 acceleration, and optimization methods (Golden Section Search, Newton's optimization, and Gradient Descent). Each method was implemented from first principles under a uniform experimental framework employing IEEE 754 double-precision arithmetic, a strict triple convergence criterion, and five complementary performance metrics. The methods were evaluated on six benchmark test functions chosen to span polynomial, transcendental, and stiff problem types, and on three real-world engineering case studies drawn from fluid mechanics (the Colebrook-White friction-factor equation), chemical engineering (Haber-Bosch ammonia synthesis equilibrium), and structural mechanics (two-bar truss optimization). The results confirm and quantify several fundamental principles. Newton-Raphson achieved quadratic convergence, reaching a tolerance of 10^{-8} in four to six iterations versus twenty-seven to twenty-eight for Bisection, while the Secant method delivered the highest efficiency index per function evaluation. A fundamental tension between convergence speed and robustness emerged consistently: the fastest methods were the least reliable from poor initial estimates, whereas the most robust methods converged slowly. Aitken acceleration reduced fixed-point iteration counts by roughly half at negligible cost. The engineering case studies validated these findings under practical conditions, with derivative-based methods yielding speedups of up to two orders of magnitude. The study concludes that no single method is universally optimal and offers an evidence-based decision framework, supported by a failure-mode analysis, to guide method selection according to problem smoothness, derivative availability, accuracy requirements, and the need for guaranteed convergence.

Keywords: nonlinear equations; iterative methods; Newton-Raphson method; Secant method; convergence order; efficiency index; optimization; benchmark functions; engineering applications

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1. Introduction

Nonlinear equations of the form f(x) = 0 and their close relatives, optimization problems, arise throughout the physical, biological, and engineering sciences. Because such equations generally possess no closed-form analytical solution, and because the number, location, and nature of their roots depend intricately on the structure of the underlying function, practitioners must rely on iterative numerical methods to obtain accurate approximate solutions. The reliability, efficiency, and accuracy of these methods determine the feasibility of large-scale simulation and design across disciplines ranging from hydraulic network analysis to chemical process engineering and structural optimization. Selecting an appropriate method for a given problem is therefore a question of considerable practical importance.

A central difficulty is that the many available methods embody different trade-offs. Bracketing methods such as Bisection guarantee convergence but only at a slow, linear rate; open methods such as Newton-Raphson converge rapidly but may diverge from a poor starting point or where the derivative vanishes; fixed-point schemes depend sensitively on the chosen iteration function; and optimization methods differ in their reliance on derivative information and their behavior on non-convex landscapes. These trade-offs are well known qualitatively, but their quantitative magnitude, and the conditions under which one method should be preferred over another, are best established through controlled experimental comparison under uniform conditions.

This paper reports such a comparison. Seven numerical methods, chosen to represent the four principal algorithmic families, were implemented from first principles and evaluated on a common suite of six benchmark functions and three engineering case studies under an identical experimental protocol. The objective is to characterize each method's convergence behavior, computational cost, accuracy, robustness, and failure modes, and to synthesize the findings into an evidence-based framework for method selection. The contribution is not the proposal of a new algorithm but a rigorous, reproducible assessment of established techniques that clarifies the practical consequences of their theoretical differences. The paper is organized as follows. Section 2 reviews the theoretical foundations and related work. Section 3 describes the research methodology, including the methods studied, the benchmark and engineering test problems, the computational environment, and the convergence and performance protocols. Section 4 presents the experimental results for each method family and for the engineering applications. Section 5 offers a comparative discussion, including a unified performance scorecard, a method-selection decision framework, a conditioning analysis, and a failure-mode catalogue. Section 6 concludes and identifies directions for future work.

2. Theoretical Background and Related Work

The root-finding problem rests on three classical existence and convergence results. The Intermediate Value Theorem guarantees a root in an interval [a, b] whenever f is continuous and $f(a)\cdot f(b) < 0$, providing the theoretical basis for all bracketing methods. The Banach fixed-point theorem guarantees that the iteration $x_{n+1} = g(x_n)$ converges to a unique fixed point when g maps an interval into itself and is a contraction, with $|g'(x)| \leq k < 1$. The Newton-

Kantorovich theorem provides conditions on f, its derivative, and the initial guess under which Newton-Raphson converges, together with an error bound at each step.

Two concepts unify the comparison of iterative methods. The first is the order of convergence p, defined by the limit of $|e_{n+1}|/|e_n|^p$ approaching a finite positive constant C as the error $e_n = x_n - x^*$ tends to zero. Linear convergence ($p = 1$) gains a fixed number of digits per step, whereas quadratic convergence ($p = 2$) approximately doubles the number of correct digits each iteration. The second is the efficiency index $E = p^{1/W}$, which combines the convergence order with the work W per iteration measured in function evaluations. For the principal open methods, Bisection gives $E = 1.000$, Newton-Raphson gives $E = 2^{1/2} \approx 1.414$, and the Secant method gives $E = 1.618^{1/1} \approx 1.618$, indicating that the Secant method is the most efficient per function evaluation among the three — a prediction the present results confirm.

The literature on these methods is extensive and continues to develop. Classical treatments establish the convergence theory and benchmark practice that frame any comparative study, and the design of benchmark suites for numerical methods follows a long tradition in the optimization literature. More recent research has pursued ever higher orders of convergence achieved with fewer function evaluations, the avoidance of second-derivative computation through finite-difference approximations, and the analysis of basins of attraction as a discriminating measure of method quality beyond convergence order alone. In parallel, optimization research, particularly in large-scale and machine-learning settings, has produced adaptive gradient methods and hybrid strategies that balance the reliability of bracketing approaches with the speed of open methods. The present study situates a controlled experimental comparison of the foundational single-variable methods within this broader context, providing the empirical grounding on which more advanced multivariate and stochastic techniques ultimately rest.

3. Research Methodology

The investigation is applied computational research employing a comparative experimental design. Rather than examining a single method in depth, it places multiple algorithms in direct competition across identical problem sets under identical computational conditions, which is the most informative approach for generating generalizable method-selection guidance. The approach combines a deductive component, in which established theory yields testable predictions for each algorithm's behavior, with an inductive component that synthesizes the empirical results across all methods and problems into broader conclusions.

3.1 Numerical Methods Studied

Seven methods were selected to provide representative coverage of the four principal algorithmic families for nonlinear equations and optimization: bracketing (interval-based) methods, open (derivative-based or derivative-free iterative) methods, fixed-point iteration methods, and function-minimization methods. Within each family, methods were chosen to span the range from simplest to most sophisticated and from no-derivative to full-derivative requirements, so that the comparison covers the full algorithmic landscape encountered in practice. Table 1 summarizes the selected methods and their theoretical properties.

Table 1. Classification of the seven numerical methods studied.

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Category	Method	Convergence Order	Derivative Req.	Guaranteed Conv.?
Bracketing	Bisection	Linear (p=1)	None	Yes (IVT)
Bracketing	Regula-Falsi	Superlinear*	None	Yes (IVT)
Open	Newton-Raphson	Quadratic (p=2)	f and f′	No (local)
Open	Secant	Superlinear (p≈1.618)	f only (2 pts)	No (local)
Fixed-Point	Fixed-Point + Aitken Δ^2	Linear → Superlinear	g(x) only	Conditional
Optimization	Golden Section Search	Linear (ratio φ)	None	Yes (unimodal)
Optimization	Newton Optim. / Grad. Descent	Quadratic / Linear	f′,f″ / f′	No / Yes (convex)

*Regula-Falsi convergence order is problem-dependent owing to one-sided stagnation; it is superlinear early but may degrade to linear under severe stagnation.

Several modified variants were also implemented to characterize each family more fully: the Illinois modification of Regula-Falsi, a modified Newton-Raphson scheme for multiple roots, six distinct iteration functions for fixed-point iteration, and five gradient-descent variants including fixed learning rate, Armijo and Wolfe line searches, momentum, and the Adam adaptive method.

3.2 Benchmark Test Functions

Six benchmark functions were selected to satisfy five criteria: mathematical diversity, known exact solutions for precise error computation, varied numerical difficulty, engineering relevance, and precedent in the numerical-analysis literature. The functions span polynomial, exponential, trigonometric, and compound transcendental categories, and all are infinitely differentiable on their test domains. Table 2 specifies each function.

Table 2. Specification of the six benchmark test functions.

ID	Function f(x)	Domain	Root(s) x*	Primary Challenge
f_1	$x^3 - 2x - 5$	[1, 3]	2.094551	Wallis cubic benchmark
f_2	$e^x - 3x$	[0, 2]	0.619061, 1.512136	Two roots; flat region at ln(3)
f_3	$x \cdot \sin(x) - 1$	[0.5, 2.5]	1.114157	Polynomial × trig product
f_4	$x^4 - x - 10$	[1, 3]	1.855806	Large f″; high Newton constant
f_5	$\cos(x) - x \cdot e^x$	[0, 1]	0.517757	Stiff exponential growth
f_6	$x^2 - 2$	[1, 2]	1.414214 (√2)	Exact Babylonian benchmark

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Two optimization benchmarks were added: a one-dimensional quartic $f(x) = x^4 - 14x^3 + 60x^2 - 70x$ on [0, 2] with an analytically computable minimum near $x^* \approx 0.743$, and a two-dimensional non-convex surface used to visualize gradient-descent trajectories.

3.3 Computational Environment and Convergence Protocol

All experiments used IEEE 754 double-precision binary floating-point arithmetic, providing approximately 15.9 significant decimal digits with unit roundoff $u \approx 1.11 \times 10^{-16}$. No external numerical libraries were used for root-finding or optimization, ensuring that observed behavior reflects the canonical algorithms rather than library-specific safeguards. Elementary functions were evaluated using standard mathematical routines correctly rounded to within one unit in the last place.

To avoid the pitfalls of single-criterion termination, convergence was declared only when a strict triple criterion was satisfied simultaneously in the same iteration: a successive-change criterion $|x_{n+1} - x_n| < 10^{-8}$; a function-residual criterion $|f(x_n)| < 10^{-10}$; and a relative-change criterion $|x_{n+1} - x_n| / \max(1, |x_n|) < 10^{-8}$. A maximum of 500 iterations was imposed for root-finding and 10,000 for gradient descent. This conservative protocol eliminates the false-convergence declarations that affect single-criterion studies.

Five performance metrics were recorded for every algorithm-function combination: iteration count, final absolute error against the known root, function-evaluation count, derivative-evaluation count, and empirical convergence order estimated from consecutive error values in the asymptotic regime. A normalized evaluation count combined function and derivative evaluations using a weight w (default $w = 2$ for the relative cost of a derivative evaluation), and the qualitative ordering of methods was verified to be insensitive to w over the range 1 to 5.

3.4 Engineering Case Studies and Validation

Three engineering case studies tested whether the benchmark findings transfer to practical problems. The first solves the implicit Colebrook-White equation for the Darcy friction factor across smooth, commercial-steel, and rough pipe scenarios spanning turbulent flow regimes. The second solves the Haber-Bosch ammonia-synthesis equilibrium for the extent of reaction at three industrially relevant temperatures. The third optimizes a symmetric two-bar truss under a vertical load, a problem with an exact analytical solution that permits precise error assessment. Algorithm implementations were verified by hand-calculation cross-checking and reproduction of the Babylonian square-root sequence, and validated by agreement of empirical convergence orders with theory to within ±0.05, by comparison of computed roots against high-precision reference values, and by agreement of engineering results with published literature data. All algorithms are fully deterministic, ensuring exact reproducibility in any compliant computing environment.

4. Results

4.1 Bracketing Methods

The Bisection method exhibited the behavior predicted by its purely interval-based theory: the bracket width halved exactly each iteration, and the error bound $(b - a)/2^n$ was matched precisely. For an initial interval of width 2 and a tolerance of 10^{-8}, the method required 28

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iterations, in close agreement with the predicted $\lceil \log_2(2\times10^8) \rceil = 28$. Across all benchmark functions the iteration count was essentially constant at 27–28, confirming that Bisection performance depends only on the initial interval width and target tolerance, not on any property of the function. The Regula-Falsi method improved on this for symmetric problems but suffered one-sided stagnation on functions of fixed concavity, ranging from 13 iterations on f_6 to 25 on f_1; the Illinois modification restored balanced bracket contraction in the stagnating cases.

4.2 Open Methods

Newton-Raphson achieved quadratic convergence on every benchmark, with the empirically estimated order stabilizing at $\hat{p} \approx 1.99–2.00$ in the asymptotic regime and convergence to machine precision in four to six iterations. Table 3 summarizes its performance. The method's sensitivity to the starting estimate was also documented: for f_2, which has two roots and a vanishing derivative at $x = \ln(3) \approx 1.099$, starting points in [1.2, 2.0] produced divergence and a start exactly at $\ln(3)$ produced a divide-by-zero failure, illustrating the necessity of a good initial estimate for open methods.

Table 3. Newton-Raphson performance across all benchmark functions.

Function	x_0	Root x*	Iterations	Final $\lvert e_n \rvert$	Conv. Order p
f_1: x^3-2x-5	2.5	2.094551	5	$< 10^{-14}$	2.00
f_2: e^x-3x (r_1)	0.5	0.619061	5	$< 10^{-14}$	2.00
f_2: e^x-3x (r_2)	1.5	1.512136	4	$< 10^{-14}$	2.00
f_3: $x\cdot\sin(x)-1$	1.0	1.114157	6	1.2×10^{-12}	2.00
f_4: x^4-x-10	2.0	1.855806	6	3.1×10^{-13}	1.99
f_5: $\cos(x)-xe^x$	0.5	0.517757	5	$< 10^{-14}$	2.00
f_6: x^2-2	1.5	1.414214	4	$< 10^{-14}$	2.00

The Secant method converged to machine precision from a poor starting pair in eight iterations and nine function evaluations on f_1, with the estimated order stabilizing at the golden ratio $\hat{p} \approx 1.618$. Because it requires only one new function evaluation per iteration and no derivative, its efficiency index of 1.618 exceeds Newton-Raphson's 1.414, and its average normalized evaluation count across benchmarks (19.0) was lower than Newton-Raphson's (23.3), making it the preferred choice when derivative evaluation is comparable in cost to function evaluation.

4.3 Fixed-Point Iteration and Acceleration

Fixed-point iteration confirmed the central role of the iteration function: convergence occurred only when $|g'(x^*)| < 1$, and the rate degraded as this contraction factor approached unity. Aitken Δ^2 acceleration applied to a fixed-point iteration with $|g'(x^*)| = 0.172$ reduced the iteration count from 12 to 6, a fifty-percent reduction achieved with no additional function evaluations and trivial implementation overhead, demonstrating that sequence acceleration is among the most cost-effective improvements available in numerical computation.

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4.4 Optimization Methods

The optimization comparison reinforced the convergence-order findings. Newton's optimization method, using curvature information, converged in a handful of steps along nearly straight-line paths toward the minimum, whereas gradient descent followed the characteristic zig-zag trajectory of steepest descent on poorly conditioned objectives. Gradient-descent performance was highly sensitive to the learning rate, with an instability threshold at $\alpha = 2/L$: the best fixed rate converged in 71 iterations, a rate one hundred times smaller failed to converge within 10,000 iterations, and a rate sixteen percent too large diverged. The Adam adaptive method identified a near-optimal step automatically and converged in 38 iterations without manual tuning, confirming the practical superiority of adaptive methods over fixed-rate gradient descent.

4.5 Engineering Case Studies

In the Colebrook-White friction-factor problem, Newton-Raphson achieved errors below 10^{-12} in four iterations across all three pipe scenarios, versus 27 for Bisection — a sevenfold reduction that, in pipeline-network simulators solving thousands of such equations, translates into substantial wall-clock savings. The Haber-Bosch equilibrium problem saw all methods converge to the same physical equilibrium at each temperature, validating correctness; Newton-Raphson required six to seven iterations across the full range of equilibrium constants spanning two orders of magnitude, while unaccelerated fixed-point iteration required 44. In the two-bar truss optimization, Newton's method applied to the KKT conditions reached the analytically exact optimum in 12 iterations, against 3,412 for gradient descent with a penalty function — a difference consistent with the qualitative gap between quadratic and linear convergence, and one in which the exterior penalty method also incurred a small constraint-satisfaction error.

Table 4. Two-bar truss optimization: four methods versus the analytical solution.

Method	Optimal θ* (°)	Min Weight W* (kg)	Iterations	Error vs Analytical
Analytical (benchmark)	35.264	48.73	—	—
Newton KKT	35.264	48.73	12	$< 10^{-10}$
Golden Section (1D)	35.263	48.73	152 / loop	$< 10^{-4}$
Gradient Descent (penalty)	35.212	48.76	3,412	0.03% subopt.
Bisection on θ	35.264	48.73	28 / step	$< 10^{-6}$

5. Comparative Discussion

Synthesizing the individual results yields a unified picture of the design space. A performance scorecard rating each method on seven criteria — convergence rate, robustness, derivative requirement, cost per iteration, accuracy, stiffness handling, and implementation complexity — shows that no method dominates across all criteria. Bisection and Regula-Falsi scored highest overall, owing to exceptional robustness, zero derivative requirements, and simplicity, but their low convergence rates limit efficiency at high accuracy. Newton-Raphson and Newton optimization scored lowest in robustness and derivative requirement, reflecting their

sensitivity to initial estimates and the cost of derivatives, even as they achieved the highest accuracy with the fewest iterations. The Secant method achieved the best balance among open methods, and fixed-rate gradient descent scored lowest overall.

The most important cross-cutting finding is that convergence speed and robustness are in fundamental tension. The highest-order methods were the least reliable from poor starting points, while the most robust methods converged slowly. This is not merely a practical limitation but a mathematical necessity: faster methods extract more information from each function evaluation and therefore depend more strongly on the quality of that information, namely the starting estimate. Hybrid strategies navigate this tension most effectively, beginning with a bracketing method to establish a safe interval and switching to an open method once the iterate enters the rapid-convergence basin; production libraries implementing such hybrids achieve the reliability of bisection together with the efficiency of open methods.

These observations support an evidence-based decision framework. The first question is whether a guaranteed bracket is available; if so and speed is non-critical, a bracketing method is appropriate. If the function is smooth and derivatives are available with a good initial estimate, Newton-Raphson minimizes iterations and maximizes accuracy; if derivatives are unavailable or costly, the Secant method offers the best derivative-free option for smooth functions. For optimization, Newton-based methods dominate smooth problems with available derivatives, Golden Section Search suits unimodal one-dimensional problems without derivatives, and adaptive gradient methods are preferred for large-scale problems where linear convergence is the acceptable cost of scalability. A conditioning analysis further showed that all benchmark functions were well conditioned, with the most challenging case being f_4, whose large second derivative produced a Newton convergence constant exceeding unity and thus required a good initial estimate to enter the convergence basin. A systematic failure-mode analysis catalogued the characteristic failures of each method — divide-by-zero, divergence, cycling, stagnation, and timeout — together with detection indicators and mitigation strategies, reinforcing the value of building failure monitoring into every iteration loop.

6. Conclusion and Future Work

This study presented a controlled, reproducible comparison of seven numerical methods for solving nonlinear equations and optimization problems, evaluated on six benchmark functions and three engineering case studies under a uniform experimental framework. The principal conclusions are clear and mutually reinforcing. Convergence order is the primary determinant of iteration count, with Newton-Raphson's quadratic convergence reducing iterations by a factor of five to seven relative to Bisection. The Secant method achieves the best efficiency index per function evaluation. Method robustness and convergence efficiency are fundamentally in tension, so that no single method is universally optimal. Aitken acceleration delivers substantial efficiency gains at negligible cost. The engineering case studies confirmed these findings under practical conditions, with derivative-based methods yielding speedups of up to two orders of magnitude. Taken together, the results provide

practitioners with a quantitative foundation for method selection: for most smooth engineering problems with available derivatives, Newton-Raphson or the Secant method offers the best accuracy-efficiency balance; where reliability is paramount, bracketing and hybrid strategies are unmatched; and for optimization, Newton-based methods dominate smooth problems while derivative-free methods remain indispensable for expensive black-box simulations.

Several limitations bound these conclusions and point toward future work. The study was confined to single-variable problems and two-variable optimization, with all test functions infinitely differentiable and free of noise. Extending the comparison to systems of nonlinear equations through multivariate Newton and quasi-Newton methods, to non-smooth and noisy functions arising from numerical simulation, and to global optimization of non-convex multimodal objectives would substantially broaden its applicability. Further valuable directions include the systematic study of hybrid and adaptive switching strategies, the integration of higher-order curvature information into large-scale stochastic optimization, and the rigorous treatment of conditioning and stability for ill-posed problems. Pursuing these directions would connect the foundational single-variable methods examined here to the multivariate, stochastic, and data-driven techniques that dominate contemporary computational practice.

References

1. Atkinson, K. E. (1989). An introduction to numerical analysis (2nd ed.). John Wiley & Sons.
2. Burden, R. L., & Faires, J. D. (2011). Numerical analysis (9th ed.). Brooks/Cole, Cengage Learning.
3. Dolan, E. D., & Moré, J. J. (2002). Benchmarking optimization software with performance profiles. Mathematical Programming, 91(2), 201–213. https://doi.org/10.1007/s101070100263
4. Kansal, M., Cordero, A., Bhalla, S., & Torregrosa, J. R. (2021). New fourth- and sixth-order classes of iterative methods for solving systems of nonlinear equations and their stability analysis. Numerical Algorithms, 87, 1017–1060. https://doi.org/10.1007/s11075-020-00997-4
5. Kincaid, D., & Cheney, W. (2002). Numerical analysis: Mathematics of scientific computing (3rd ed.). Brooks/Cole.
6. Moody, L. F. (1944). Friction factors for pipe flow. Transactions of the ASME, 66(8), 671–684.
7. Moré, J. J., Garbow, B. S., & Hillstrom, K. E. (1981). Testing unconstrained optimization software. ACM Transactions on Mathematical Software, 7(1), 17–41. https://doi.org/10.1145/355934.355936
8. Nadeem, G. A., Aslam, W., & Ali, F. (2023). An optimal fourth-order second-derivative-free iterative method for nonlinear scientific equations. Kuwait Journal of Science, 50(4). https://doi.org/10.48129/kjs.18253

9. Sana, G., Mohammed, P. O., Al-Sarairah, E., & Noor, M. A. (2020). Iterative methods for solving nonlinear equations using quadrature and decomposition. Mathematical Statistician and Engineering Applications.
10. Smith, J. M., Van Ness, H. C., & Abbott, M. M. (2004). Introduction to chemical engineering thermodynamics (7th ed.). McGraw-Hill.
11. Stokes, D. E. (1997). Pasteur's quadrant: Basic science and technological innovation. Brookings Institution Press.
12. Thota, S., & Shanmugasundaram, P. (2022). On new sixth and seventh order iterative methods for solving nonlinear equations using homotopy perturbation technique. BMC Research Notes, 15(1), 267. https://doi.org/10.1186/s13104-022-06154-5
13. Traoré, C., & Pauwels, E. (2021). Sequential convergence of AdaGrad algorithm for smooth convex optimization. Operations Research Letters, 49(4), 452–458. https://doi.org/10.1016/j.orl.2021.04.011
14. Usman, M., Iqbal, J., Khan, A., Ullah, I., Khan, H., & Alzabut, J. (2023). A new iterative multi-step method for solving nonlinear equations. MethodsX, 14, 103394. https://doi.org/10.1016/j.mex.2023.103394
15. Xu, D., Zhang, S., Zhang, H., & Mandic, D. P. (2021). Convergence of the RMSProp deep learning method with penalty for nonconvex optimization. Neural Networks, 139, 17–23. https://doi.org/10.1016/j.neunet.2021.02.011