LDPC Codes for the Two-User GMAC

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Abstract—**The capacity region of Gaussian Multiple Access Channels (GMACs) has been known since 1971. The efficient design of powerful codes that can achieve points near the dominant face of the capacity region, where the sum-rate is maximal, is an interesting problem. Significant progress has been made in this direction using time sharing, rate-splitting, as well as joint iterative decoding. Joint iterative decoding seems to be the most promising path, especially for codes that have low-complexity decoders, like Low-Density Parity-Check (LDPC) codes.**

LDPC codes are capacity-approaching over a wide variety of channels. Additionally, elegant tools, such as Density Evolution and EXIT charts, can be used to accurately predict the asymptotic performance of an LDPC code ensemble. These tools can be used for the design of optimal LDPC codes, allowing for transmission over many types of channels with vanishingly small probability of error.

In this paper, we focus on the two-user GMAC. To the best of our knowledge, there exist two LDPC code design frameworks for this channel. We provide simulation results that demonstrate the excellent finite length performance of codes designed using the proposed method.

Keywords—**GMAC, LDPC.**

I. INTRODUCTION

The capacity region of a two-user GMAC is the pentagon defined by

$$R^{[1]} \leq C\left(\frac{P_1}{N}\right), \qquad R^{[2]} \leq C\left(\frac{P_2}{N}\right),$$

$$R^{[1]} + R^{[2]} \leq C\left(\frac{P_1+P_2}{N}\right),$$

Where $C(x) = \frac{1}{2}log_2(1+x)$, P_i is the power of user i, i = 1, 2, and N is the noise power. The dominant face is the side of the pentagon where the sum-rate is maximum. Its endpoints can be achieved by successive decoding; the remaining points can be achieved either by time-sharing, rate splitting, or joint decoding. A framework for LDPC code design for the two-user GMAC, using joint decoding, appears in, which, however, is limited to the case where the users have equal transmit powers. In this work, we generalize this framework and present an EXIT chart based LDPC code design for the unequal power two-user GMAC.

II. SYSTEM MODEL AND PRELIMINARIES

A. System Model

Let the length-n codeword's of users 1 and 2 be $c^{[1]} \in C_1$ and $c^{[2]} \in C_2$, respectively, where C_1 and C_2 denote the respective codebooks. In the low SNR regime, the sum rate is near unity and, thus, binary codes suffice. The BPSK modulated codeword's are $x^{[j]} = 1-2c^{[j]}$; j = 1; 2. Let P_1 and P_2 denote the users' normalized, with respect to the noise variance, transmit powers. Then, the output of the GMAC channel is

$$y = \sqrt{P_1}\,x^{[1]} + \sqrt{P_2}\,x^{[2]} + w, \quad w \sim N(0,1)$$

Since the noise has no unit variance, we define $SNR_1 = P_1$ and $SNR_2 = P_2$.The following symmetry property, which will be used later, Can be easily verified.

$$p\left(y_i \middle| x_i^{[1]}, x_i^{[2]}\right) = p\left(-y_i \middle| -x_i^{[1]}, -x_i^{[2]}\right).$$

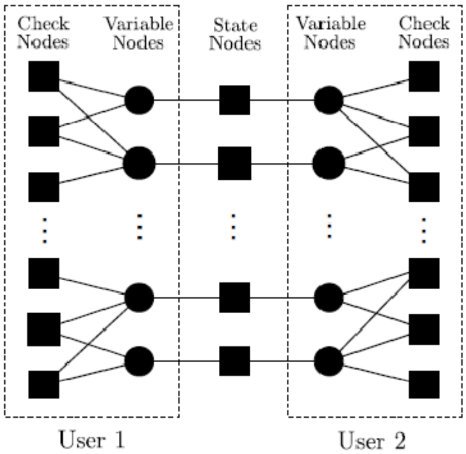


Fig1. Example of a Tanner graph representing the bit-wise MAP decoding rule for a two-user MAC.

B. LDPC Codes

An ensemble of LDPC codes is described by its edge perspective variable and check node degree distributions, λ(x) and ρ(x), respectively, where

$$\lambda(x) = \sum_i \lambda_i x^{i-1} \ , \rho(x) = \sum_i \rho_i x^{i-1}$$

The corresponding node perspective variable and check node degree distributions are

$$L(x) = \frac{\int_0^x \lambda(z)\,dz}{\int_0^1 \lambda(z)\,dz}, R(x) = \frac{\int_0^x \rho(z)\,dz}{\int_0^1 \rho(z)\,dz}$$

The design rate of an LDPC code is defined as

$$r = 1 - \frac{\sum_i \rho_i/i}{\sum_i \lambda_i/i}$$

III. BIT-WISE MAP DECODING AND BELIEF PROPAGATION

A. Bit-wise MAP Decoding

Given the output vector y, $\hat{x}_i^{[1]}$ denotes the MAP estimate for codeword bit i of user 1, i.e.

$$\hat{x}_i^{[1]} = \arg\max_{x_i^{[1]}} \sum_{\sim x_i^{[1]}} \sum_{x^{[2]}} (\prod_{k=1}^{n} p(y_k | x_k^{[1]}, x_k^{[2]})) 1_{C_1(c^{[1]})} 1_{C_2(c^{[2]})}$$

where the first sum is over all elements of $x^{[1]}$ except $x_i^{[1]}$ and 1A(.) is the indicator function of the set A. An analogous expression holds for $\hat{x}_i^{[2]}$. The code can be described by the Tanner graph depicted in Fig1. Belief Propagation (BP) can be used for the reliable and efficient decoding provided that the corresponding graph is asymptotically cycle-free.

B. Belief Propagation

All messages are assumed to be in Log-Likelihood Ratio form. Let $cv_k^{[j]}$ and $vc_k^{[j]}$ denote, respectively, the check-to variable and variable-to-check messages for check and variable node k of user j. These messages follow single-user BP rules. Let $vs_k^{[j]}$ denote the variable-to-state message from user j towards state node k and let $sv_k^{[j]}$ denote the state-to-variable message from state node k towards user j. Messages $vs_k^{[j]}$ follow single-user BP rules. However, the update rule for $sv_k^{[1]}$ needs message $vs_k^{[2]}$ and the update rule for $sv_k^{[2]}$ needs message $vs_k^{[1]}$. More specifically, using standard function node message-passing rules, we obtain

$$sv^{[1]} = log\frac{e^{-\frac{(y-\sqrt{p_1}-\sqrt{p_2})^2}{2}} e^{vs^{[2]}} + e^{-\frac{(y-\sqrt{p_1}+\sqrt{p_2})^2}{2}}}{e^{-\frac{(y+\sqrt{p_1}-\sqrt{p_2})^2}{2}} e^{vs^{[2]}} + e^{-\frac{(y+\sqrt{p_1}+\sqrt{p_2})^2}{2}}}$$

$$sv^{[2]} = log\frac{e^{-\frac{(y-\sqrt{p_1}-\sqrt{p_2})^2}{2}} e^{vs^{[1]}} + e^{-\frac{(y-\sqrt{p_1}+\sqrt{p_2})^2}{2}}}{e^{-\frac{(y+\sqrt{p_1}-\sqrt{p_2})^2}{2}} e^{vs^{[1]}} + e^{-\frac{(y+\sqrt{p_1}+\sqrt{p_2})^2}{2}}}$$

It can be easily verified that

$$sv^{[1]}(-y, -vs^{[2]}) = -sv^{[1]}(y, -vs^{[2]})$$

$$sv^{[2]}(-y, -vs^{[2]}) = -sv^{[2]}(y, -vs^{[2]})$$

Thus, if the densities of the messages entering the state node are symmetric, then the densities of the messages leaving the state node will also be symmetric. In the limit of infinite block length, the overall two-user Tanner graph is cycle-free, thus tools such as Density Evolution and EXIT Charts can be used for the evaluation of the asymptotic performance of LDPC codes for the unequal power two-user GMAC. We consider a parallel schedule, i.e., we first perform a round of standard BP for each user and then update the state node messages.

IV. DENSITY EVOLUTION AND EXIT CHARTS

A. *Density Evolution*

Density Evolution tracks the densities of the messages exchanged by the BP decoder and can be used for the derivation of conditions that asymptotically guarantee vanishingly small probability of decoding error.

If we restrict ourselves to the cases where $x_i^{[1]} = x_i^{[2]}$, the GMAC channel is equivalent to a BI-AWGN channel with input power $\sqrt{p_1} + \sqrt{p_2}$,which is symmetric in the single user sense. Let $a_1(z)$ denote the density of $log\frac{p(y_i|x_i=+1)}{p(y_i|x_i=-1)}$ conditioned on $X_i = +1$ when $x_i = x_i^{[1]} = x_i^{[2]}$. We can model transmission over each of the two "restricted" channels multiplicatively as

$$Y_i^{[1]} = x_i Z_i^{[1]} \; and \; Y_i^{[2]} = x_i Z_i^{[2]}$$

Where $Z_i^{[1]}$ is distributed according to $a_1(z)$ and $Z_i^{[2]}$ is distributed according to $a_2(z)$. Since $x^{[1]}$ and $x^{[2]}$ are independent LDPC codeword's, the events $x_i^{[1]} = x_i^{[2]}$ and $x_i^{[1]} = -x_i^{[2]}$ are euiprobable. Thus, in the typical case half the state nodes will be $x_i^{[1]} = x_i^{[2]}$ nodes and half the state nodes will be $x_i^{[1]} = -x_i^{[2]}$ nodes.

Due to space limitations, we omit the density evolution expressions and proceed directly to the stability condition, Gaussian approximation and EXIT chart analysis.

B. *Stability Condition*

We observe that the receiver of an unequal power two-user MAC channel is equivalent to the best receiver of a two-user broadcast channel. Thus, the stability condition is

$$\lambda_2^{[j]} \sum_i (i-1)\rho_i^j < \exp\left(\frac{p_j}{2}\right) , j = 1,2$$

C. *Gaussian Approximation*

Under the Gaussian Approximation (GA), all message densities are approximated as symmetric Gaussian. The variable-to-check and check-to-variable messages follow standard GA rules. The variable-to-state messages also follow standard GA rules with the difference that averaging is done over L(x), instead of $\lambda(x)$. We assume that the codeword of user 1 is the all-one codeword, while the codeword of user 2is a codeword of type one-half. If we further assume that the variable-to-state messages are symmetric Gaussian with mean μ, Functions $F_+^{[1]}$and $F_-^{[1]}$ (resp. $F_+^{[2]}$and $F_-^{[2]}$) are the means of the state-to-variable messages towards user 1 (resp. user 2) from state nodes that are connected to a +1 and a-1 variable node of user 2 (resp. user 1), respectively.

D. *EXIT Charts*

Extrinsic Information Transfer (EXIT) charts can be used for the accurate analysis of the behavior of LDPC decoders. A metric that is commonly used in EXIT chart analyses is the mutual information between the messages exchanged by the decoder and the codeword bits.

Let $I_{AV}^{[j]}$ (resp. $I_{ES}^{[j]}$) denote the mutual information between the codeword bits and the check-to-variable (resp. state-to variable) messages of user j. We can show that the EXIT chart $I_{EVC}^{i,[j]}$ describing the variable to- check messages emanating from a variable node of degree i is

$$I_{EVC}^{i,[j]}\left(I_{AV}^{[j]}, I_{ES}^{[j]}\right) = J\left(\sqrt{(i-1)J^{-1}(I_{AV}^{[j]})^2 + J^{-1}(I_{ES}^{[j]})^2}\right)$$

Where

$$J(\sigma) \coloneqq 1 - \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\sigma^2/2)^2}{2\sigma^2}} \log_2(1+e^{-x})\,dx$$

Averaging over λ(x), we get the variable – to –check EXIT chart

$$I_{EVC}^{[j]}\left(I_{AV}^{[j]}, I_{ES}^{[j]}\right) = \sum_i \lambda_i I_{EVC}^{i,[j]}\left(I_{AV}^{[j]}, I_{ES}^{[j]}\right)$$

Similarly, it can be shown that the EXIT chart $I_{EVC}^{[j]}$ decribe in the variable – to- state message is

$$I_{EVC}^{[j]}\left(I_{AV}^{[j]}\right) = \sum_i L_i J\left(\sqrt{i}\,J^{-1}(I_{AV}^{[j]})\right)$$

By exploiting the duality between the check and variable nodes, it can be shown that the EXIT chart describing the check –to-variable message can be well approximated as

$$I_{EC}^{[j]}\left(I_{AC}^{[j]}\right)=\sum_{i}\rho_i I_{EC}^{i,[j]}\left(I_{Ac}^{[j]}\right)$$

We now need to derive the EXIT function for the state-to-variable messages. We first briefly present the existing results for the equal power case [12] and we will then proceed with the EXIT functions for the unequal power case.

V. OPTIMIZATION PROCEDURE

As we have already mentioned, a sufficient condition for perfect decoding is that the mutual information between the variable node output messages and the codeword bits is larger than the mutual information between the variable node input messages and the codeword bits . Since $I_{EC} = I_{AV}$, an equivalent condition is that the inverse of I_{EC} lies strictly below I_{EVC} for both users. Furthermore, it has been shown that it is reasonable to constrain ρ (x) to be concentrated i.e. $\rho(x)=x^k\ k\in N$ lead to efficient singleuser design , we assume $\rho^{[j]}(x)=x^{k_j}, k_j\in N$. Then $I_{EC}^{[j]}\left(I_{AC}^{[j]}\right)=I_{EC}^{k,[j]}\left(I_{AC}^{[j]}\right)$ and the inverse $I_{EC}^{[j]}\left(I_{AC}^{[j]}\right)$, denoted $I_{AC}^{[j]}\left(I_{EC}^{[j]}\right)$ can be approximated as

$$I_{AC}^{[j]}\left(I_{EC}^{[j]}\right)\approx 1-J\left(\frac{1}{\sqrt{k_j-1}}J^{-1}(1-I_{EC}^{[j]}\right)$$

This simplification is crucial since the optimization problems which we will state in the sequel would otherwise be non-linear with respect to the coefficients of $\lambda^{[j]}(x)$. This assumption also simplifies the stability conditions and becomes

$\lambda_2^{[j]}<\frac{\exp\left(\frac{P_j}{2}\right)}{(k_j-1)}, j=1,2.$ respectively.

VI. RESULTS

We set σ= 0:7945 which corresponds to a sum-rate of 1, i.e. r = 0:5 for each user. Optimized degree distributions for the equal power case and for various values of v_{max} can be seen in Table 1. We observe that when we increase v_{max}, the sum-rate increases and for v_{max} = 200 we are only 0:0176 bits per channel use away from the maximal sum-rate. In order to assess the finite length performance of the optimized ensembles, we created random codes of length n = 10^5 for each user with no cycle removal. The maximum number of BP decoding iterations was set to 200. For the code of the user who transmits the codeword of type one-half, we had to create a specific encoder. Thus, a structure which allows for efficient encoding was enforced for this code. Since both codes belong to the same ensemble, their performance is close to the ensemble average with probability that rises exponentially to 1 in the block length. So, in Fig.2 we only plot the performance of one code for the sake of simplicity.

$v_{\max}$	50	100	200
λ_2	0.3098	0.2553	0.1841
λ_3	0.3343	0.3138	0.2789
λ_{21}			0.0561
λ_{22}		0.0173	0.1592
λ_{23}		0.2340	
λ_{24}	0.1363		
λ_{25}	0.1144		
λ_{50}	0.1053		
λ_{100}		0.1796	
λ_{200}			0.3216
$\rho(x)$	x^6	x^7	x^8
Rate	0.48734	0.48977	0.49122

Table1 : Optimized degree distributions for the equal power case.

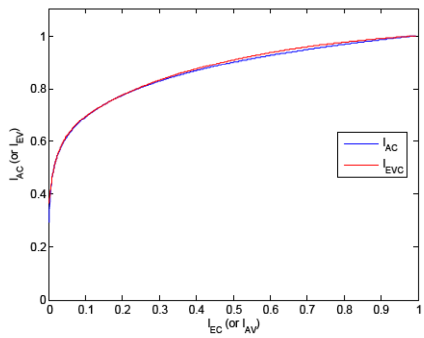
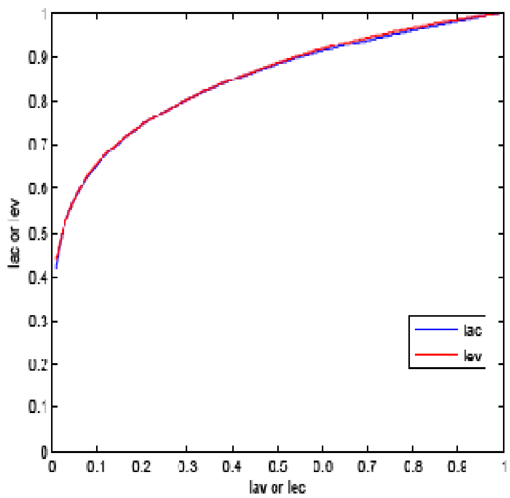


Fig2. Equal power optimization results for vmax = 100.

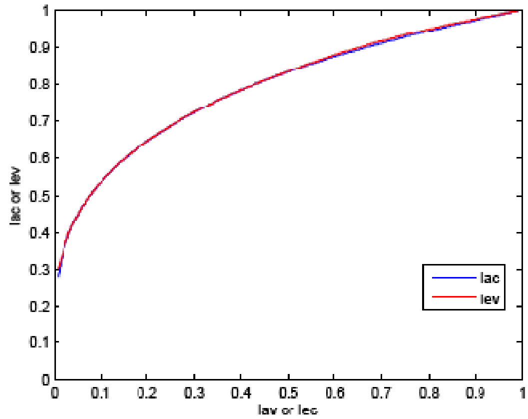
For the unequal power case, we explored two scenarios. In the first scenario P_1 and P_2 were relatively close, while, in the second scenario the gap between P_1 and P_2 was significant. Optimized degree distributions for both scenarios, both users, and for various values of v_{max} can be seen in Table2. The corresponding EXIT charts can be seen in Fig3.

	$P_1 = 1.5, P_2 = 1$				$P_1 = 1.75, P_2 = 0.75$	
$v_{\max}$	50		200		200	
	$\lambda^{[1]}(x)$	$\lambda^{[2]}(x)$	$\lambda^{[1]}(x)$	$\lambda^{[2]}(x)$	$\lambda^{[1]}(x)$	$\lambda^{[2]}(x)$
λ_2	0.2950	0.3668	0.2429	0.1853	0.1649	0.2159
λ_3	0.3766	0.3529	0.3595	0.2762	0.4754	0.2559
λ_8						0.0320
λ_9						0.1181
λ_{12}		0.0059		0.0489		
λ_{13}		0.1247		0.0705		
λ_{21}	0.3766					
λ_{22}			0.1433		0.2609	
λ_{23}			0.0800			
λ_{32}				0.0569		0.1239
λ_{33}				0.0567		0.0171
λ_{42}		0.1126				
λ_{43}		0.0370				
λ_{50}	0.1408					
λ_{56}					0.0036	
λ_{57}					0.0952	
λ_{98}			0.0631			
λ_{99}			0.1111			
λ_{200}				0.3054		0.2371
	$\rho^{[1]} = x^6$	$\rho^{[1]} = x^4$	$\rho^{[1]} = x^7$	$\rho^{[2]} = x^7$	$\rho^{[1]} = x^{10}$	$\rho^{[2]} = x^5$
Rate	0.4984	0.3644	0.5060	0.3726	0.5756	0.3059

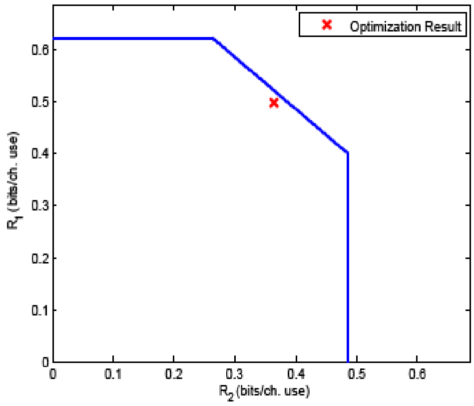
Table2: Optimized degree distributions for the unequal power case



(a) EXIT chart for user 1.



(b) EXIT chart for user 2.



(c) Rate pair and capacity region.

Fig.3 Optimization results for P1 = 1:5 and P2 = 1 and vmax = 200.

VII. CONCLUSIONS

We developed a framework for the optimization of LDPC codes for transmission over the two-user GMAC. We saw that, under some assumptions, the optimization problem can be expressed as an alternating sequence of linear programming problems, which can be solved efficiently. The resulting codes are close to optimal, in terms of sum-rate, and exhibit very good finite-length behavior.

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