Topological Conjugacy of Logistic Map

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Abstract : Topological conjugacies preserve many topological dynamical properties. Thus, if we find a topological conjugacy of a map f with a simpler map g , we can analyse the simpler map g to obtain information about dynamical properties of the original map .In this paper We have established a topological conjugacy of logistic map $L_\mu(x)= \mu x(1-x)$,x ∈ [0,1] with the quadratic map $Q(x)=x^2+c$,tent map T: [0,1]->[0,1] ,T(x) =2x, if 0<=x<=1/2 and T (x)=2(1-x) ,if ½<x<=1 ;and logistic type map $F_\mu(x)=(2- \mu)x(1-x)$, x ∈ [0,1]

Keywords: Logistic map ,Topological conjugacy, Topological transitivity ,Homeomorphisim.

1.1.Introduction

By $f(x) = \mu x(1-x)$ Logistic Map is defined, where μ is the parameter.For decades,several iterated functions have been extensively studied ,and rich contents bhave een explored .Logistic map is one of the well –known maps and has became a standared map for studying iteration .This map contains all the interesting subjects in non-linear dynamics; .we list some references in [1-9].In general,the values of x and m of logistic map are restricted in the range ,$0 <= x <= 1, 0 <= \mu <= 4$ so that each x in the interval [0,1] is mapped onto the same interval [0,1].It is known that there is a stable fixed point x*=0 in the range 0<=μ<=1,and another stable fixed point x*=1-1/μ in the range 1<=μ<=3.After that ,we have period-doubling bifurcation at μ=3, 3.4494897, 3.54409 .... .These numerical results are well known and are easy to reproduce on computer. However, it is a puzzle why we have two neighbour regions, 0<=x<=1 and 1<=μ<=3, that each has a stable fixed point of f. According to sharkovsky ordering [1],the appearance of the order of periods should be 1->2->4->8->..................,but instead we now have 1->1->2->4->8->..................... .This seems the Sharkovsky ordering is slightly violated. However it dose not.

1.2 .Topological conjugacy :

Definition: A map $f: X \to X$ is a homeomorphism if map $f: X \to X$ is continuous and invertible and the invers $f^{-1}: X \to X$ is continuous.

Definition: Two topological dynamical systems $f: X \to X$ and $g: Y \to Y$ are topologically conjugated if they are conjugated and the conjugacy map $h: X \to Y$ is a homeomorphism. We will call h a topological conjugacy.i.e. h∘f=g ∘h

If in the above definition we only require the map $h: X \to Y$ to be continuous, then we say than g is a factor of f .If in addition ,h is an onto map ,than we say that g is a quasi –factor of f.

Definition: Two topological dynamical systems $f: X \to X$ and $g: Y \to Y$ are topological semi – conjugated if they are semi-conjugated and semi-conjugacy map $h: X \to Y$ is not only surjective but also continuous. We call h a topological semi-conjugacy .[3,6]

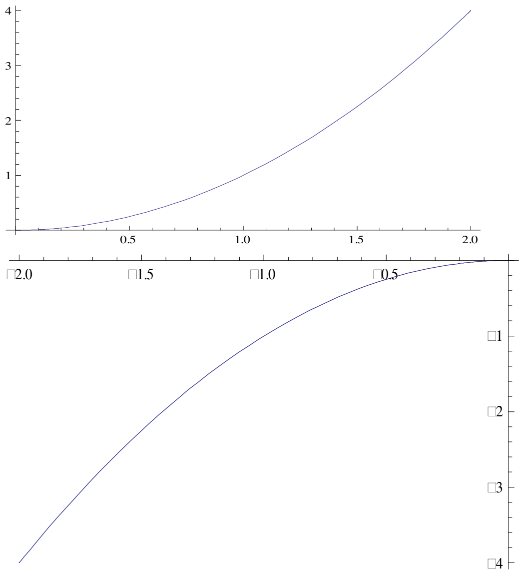
Let us define the topological conjugacy condition h ∘f=g∘h in the following figure . The idea is that both upper routes from the upper left X to the lower right Y-across the top, then down the right side , and down the left side ,then across the bottom give the same result .

We say that the diagram commutes essentially , h is mapping the function f to the function g.

	f	
X	→	X
↑	g	↓
Y	→	Y

Which shows h∘.f=g∘.h

For example the map $h: R \to R$ defined by h(x)=x^2 ,x≥o;andh(x)= $-x^2$,x≤0 ; h is a



Homeomorphisim.

1.3 Transitivity :

Some times given dynamical system $f: X \to X$,when we iterate x_0 ∈X ,the orbit $O(x_0)=\{x_0, f(x_0),......\}$, spreads itself evenly over X, so that $O(x_0)$ is a dense set in X.

Thus $f: X \to X$ is said to be topologically transitive if there exists $x_0 \in X$ such that $O(x_0)$ is a dense subset of X. A transitive point for f is a point x_0 which has a dense orbite under f . If f is transitive , then there is a dense set of transitive points , since each point in $O(x_0)$ will be a transitive point .

Proposition.1.2.1 If f : X → X and g : Y → Y are maps conjugate via a conjugacy

h : X → Y ; h• f = g• h, then

1. h $\bullet f^n = g^n \bullet h$ for all n ∈ Z+, (so f^n and g^n are also conjugate).

2. If c is a point of period m for f, then h(c) is a point of period m for g∘. c is attracting if and only if h(c) is attracting.

3. f is transitive if and only if g is transitive.

4. f has a dense set of periodic points if and only if g has a dense set of periodic points.

5. f is chaotic if and only if g is chaotic.[11,12,14]

Proof. 1. h $\bullet f^2$ = h •f• f = g •h • f = g •g •h = g^2 •h, and in the same way

h• f^3 = g^3 •h, and continuing inductively the result follows.

2. Suppose that $f^i(c) \neq c$ for 0 < i <m and $f^m(c)=c$, then h $\bullet f^{\,i}(c) \neq h(c)$ for

0 < i <m since h is one-to-one, and so g^i •h(c) ≠h(c) for 0 <i<m. In addition,

h • $f^m(c)=g^m$ •h(c), or $h(c)=g^m(h(c))$, so h(c) is a period m point for g.

We shall only show that if p is an attracting fixed point of f (so that there is an open ball B(p) such that if x ∈ B(p) then $f^n(x) \to p$ as n →∞), then h(p) is an attracting fixed point of g. Let V = h(B(p)), then since h is a homeomorphism, V is open in Y and contains h(p). Let y ∈ V , then h−1(y) ∈ B(p), so that fn(h^{-1}(y)) → p as n →∞ . Since h is continuous, $h(f^n(h^{-1}(y))) \to h(p)$ as n →∞, i.e.,

$g^n(y)=h \bullet f^n \bullet h^{-1}(y) \to h(p)$, as n →∞, so h(p) is attracting.

3. Suppose that $O(z)=\{z,f(z),f^2(z),...\}$ is dense in X and let V ⊂ Y be a non- empty open set. Then since h is a homeomorphism, $h^{-1}(V)$ is open in X, so there exists k ∈ Z+ with

$f^k(z) \in h^{-1}(V)$. It follows that $h(f^k(z)) = g^k(h(z)) \in V$, so that

$O(h(z)) = \{h(z),g(h(z)),g^2(h(z)),...\}$

is dense in Y , i.e., g is transitive. Similarly, if g is transitive, then f is transitive.

4. Suppose that f has a dense set of periodic points and let $V \subset Y$ be non-empty and open. Then $h^{-1}(V)$ is open in X, so contains periodic points of f. As in (3), we see that V contains periodic points of g. Similarly if g has a dense set of periodic points, so does f.

5. This now follows from (3) and (4).

1.4 Conjugacy and Fundamental Domains :

We have seen that two dynamical systems f and g with different dynamical proper- ties cannot be conjugate. On the other hand, sometimes we have dynamical systems having seemingly very similar dynamical properties and which we would like to show are conjugate. This is sometimes possible using the notion of fundamental domain, a set on which we construct a map h in an arbitrary manner and show that it extends to a conjugacy on the whole space. We first illustrate this idea with homeomorphisms $f,g : R \to R$. We look at a fairly straight forward case where both homeomorphisms are order preserving and have no fixed points (in fact lie strictly above the line y = x).[15]

Proposition 1.4.1 Let $f,g : R \to R$ be homeomorphisms satisfying $f(x) \to x$ and $g(x) \to x$ for all $x \in R$. Then f and g are conjugate.

Proof. The idea for the proof is a follows: Select $x0 \in R$ arbitrarily and consider the 2-sided orbit $Of(x0)=\{f^n(x_0) : n \in Z\} = \{...,x_{-1},x_0,x_1,x_2,...\}$.

Since $f(x) > x$ for all x, this is an increasing sequence: $...x_{-1} < x_0 < x_1 < x_2 <...$, so that the sets $[x_{-1},x_0),[x_0,x_1),[x_1,x_2),...,$ are disjoint and their union is all of R. We must have $\lim_{n\to\infty} x_n = \infty$ since other wise the limit would exist and would have to be a fixed point. There are no fixed points since $f(x) > x$ always. The set $I =[x_0,f(x_0)) = [x_0,x_1)$ is called a fundamental domain for f. Set $J =[x_0,g(x_0))$ and define a map $h : I \to J$ arbitrarily as a continuous bijection (e.g., we can set $h(x_0)=x_0$ and $h(f(x_0)) = g(x_0)$ and then linearly from I to J). Now every other orbit of f intertwines with $Of(x_0)$: if $y_0 \in (x_0,x_1)$, then $y_n = f_n(y_0) \in f_n(I)$, so lies between x_i and x_{i+1}. It follows that every orbit has a unique member in the interval $[x_0,x_1)$ and we use this to extend the definition of h to all of R. If $x \in f_n(I)$ we define h(x) by mapping x back to I via f_{-n}, then using $h(f^{-n}(x))$ which is well defined, and then mapping back to $g^n(J)$ using g^n. i.e., if $x \in f^n(I)$, $n \in Z$, define $h(x)=g^n \circ h \circ f^{-n}(x)$.

In this way, h is defined on all of R. We can check that h is one-to-one. It is onto because $h(f^{-n}(I)) = g^n(J))$ for each n, and we can check that it is continuous. Finally, because of the definition of h, ifx $\in R$, then $x \in f_n(I)$ for some $n \in Z$, so $x = f^n(y)$ for some $y \in I$. Then

$g \bullet h(x)=g(g^n \bullet h \bullet f^{-n}(x)) = g^{n+1} \bullet h \bullet f^{-(n+1)}(f(x)) = h \bullet f(x)$,

so that f and g are conjugate.

Our Discussion**:**

1. **$L_\mu(x)= \mu x(1-x)$,and $Q_c(x)=x^2+c$ are linearly conjugate** .

Proof: Let h(x)= ax+b ,be a linear function ,then h(x) is surjective and continuous also h is invertible and h is homeomorphism .

We now try to solve $h \bullet L_\mu(x)= Q_c \bullet h(x)$

$\Rightarrow a\,\mu x(1-x)+b =(ax+b)^2+c$

$\Rightarrow a\,\mu x- a\,\mu x^2+b = a^2 x^2+2bx+b^2+c$

Now by compairing the co –ifficents of various powers of x from both sides we have $a= -\mu$, $a\mu = 2ab \Rightarrow b = \mu/2$;

Therefore $h(x)= -\mu x+ \mu/2$ and $c=(2\mu-\mu^2)/4$

Again $h(0)= \mu/2$ and $h(1)= -\mu/2$

Therefore L_μ on the interval [0,1] is linearly conjugate to Q_c on the interval

$[-\mu/2 , \mu/2]$.

For $\mu=4$, $L_4(x)= 4x(1-x)$, on [0,1] is conjugate to $Q_c(x)=x^2+c$ on the interval [-2 ,2] when c=-2 .In particular, Q_{-2} on [-2,2] is chaotic.

2. The logistic map L_4 :[0,1]→[0,1], $L_4(x)=4x(1-x)$ is conjugate to the tent map

T: [0,1]→[0,1] ,T(x) =2x,if $0\leq x \leq 1/2$ and T(x)=2(1-x) ,if $½\leq x <1$;

Proof: Define h:[0,1]->[0,1] by $h(x)=\sin^2(\pi x/2)$.

Here h(0)=0 ,h(1)=1 and h is continuous implies h surjective ,i.e.h([0,1]) =[1,0].

Again $h(x)= (\pi/2)\sin(\pi x) >0$ if $0<x<1$. Thus h is monotonic , i.e. h ,is injective . Thus h is invertiable .$h^{-1}(x) = \frac{2}{\pi}\arcsin(\sqrt{x})$ and h is homeomorphism.

Also $L_4 h\bullet(x)=L_4(\sin^{2(}(\pi x/2))$ $=4\sin^2(\pi x/2)(1-(\sin^2(\pi x/2))=\sin^2(\pi x)$.

And $h\ T\bullet(x)=h(Tx)=h(2x)$ if $0<=x<=1/2$;and $h(Tx)= h(2-2x)$,if $½<x<=1$

From both $h\ T\bullet(x)= \sin^2(\pi x)$

Therefore $L_4\bullet h =h\bullet T$ implies that L_4 and T are conjugate .

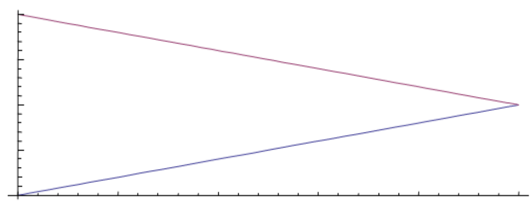


Fig1: Tent map T

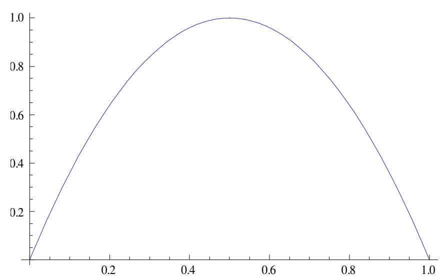


Fig 2:Logistic map L_4

Figure shows tent mapT and logistic map L_4 are topologically conjugated.

3. The logistic map $L_\mu(x)= \mu x(1-x)$,$x\in [0,1]$ is conjugate to the logistic type map

$F\mu=(2-\mu)x(1-x)$,($\mu\neq 0$),[11].

Proof: Define $h(x)=ax+b$,be a linear function ,then h(x) is surjective and continuous also h is invertible and h is homeomorphism.

Now $h \bullet L_\mu (x)= F\mu \bullet h(x)$

$=>a\{ \mu x(1-x) \}+ b =(2-\mu)(ax+b)- (2-\mu)(ax+b)^2$

$=> a \mu x - a \mu x^2+b= (2-\mu)ax -(2-\mu)b -(2-\mu)a^2x^2-2(2-\mu)axb -(2-\mu)b^2$

By compairing the various power of co –ifficient of x from both sides

$a=\mu/2-\mu$; $b=(1-\mu)/2-\mu$

Therefore $h(x)=(\mu/2-\mu)x +(1-\mu)/2-\mu$.

Again $h(0)= (1-\mu)/2-\mu$ and $h(1)=1/2- \mu$

Therefore L_μ on the interval [0,1] is linearly conjugate to $F\mu$ on the interval

.[$(1-\mu)/2-\mu$, $1/2- \mu$].

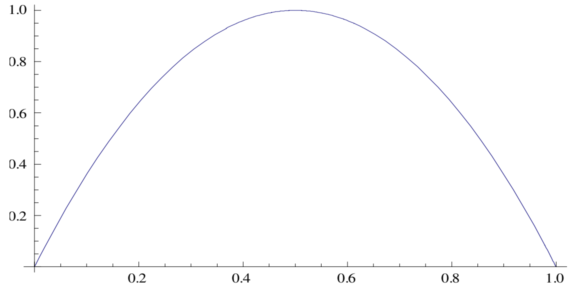


Fig.L_4

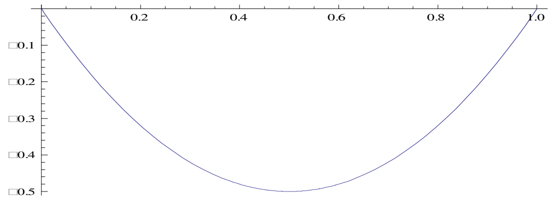


Fig3 .F_4

Figure shows logistic map L_4 and F_4 map are topologically conjugated .

4.TRANSITIVITY OF LOGISTIC MAP

For the logistic map $L_4(x)=4x(1-x)$ let us find the period n points ,such that to solve the equation $L^n_4(x)=x$ .When $x=\sin^2\theta$, this becomes

$\text{Sin}^2(\theta) = \sin^2(2^n \theta)$.

This gives rise to the two equations

$\pm\theta =2^n \theta +2k\pi$, or $\pm \theta = (2 k + 1)\pi – 2^n \theta$, for some $k \in Z$.

This can be summarized as a single equation:

$\pm\theta = 2^n\ \theta + k\pi \Rightarrow \theta = k\pi/(2^n \pm 1)$,n=1 ,2,3..., k∈ Z

so that, $Per^n(L_4)$={sin2(kπ/(2^n − 1) : 0≤ k<2^n-1}∪{$\sin^2$[kπ/ (2^n +1)] : 0<k≤ 2^n-1}. It follows that L_4 has points of all possible periods. And also the set of all periodic points constitutes a "dense" set in [0,1] but each of these points is unstable .Thus we have the logistic map is transitive for μ=4.

5.FUNDAMENTAL DOMAINS OF LOGISTIC MAP

Consider the logistic maps Lμ(x)=μx(1 − x) for various values of μ ∈ (0,4] and x ∈ [0,1]. We first show that for 0 <μ<λ≤ 1, Lμ and Lλ are conjugate. There is a slight complication here as these maps are not increasing, but they do have an attracting fixed point at 0, and we have that the basin of attraction is all of [0,1].[15]. We first deal with the interval on which the maps are increasing, [0,1/2], and look at the restriction of the functions to this interval. Our aim is to construct a homeomorphism h : [0 ,1] → [0,1] with the property Lλ • h = h • Lμ. Take Lμ(0,1/2] = (μ/4,1/2] as a fundamental domain for Lμ and Lλ(0,1/2] = (λ/4,1/2] as a fundamental domain for Lλ. Define h : (μ/4,1/2] → (λ/4,1/2] by h(1/2) = 1/2 and h(μ/4) = λ/4 and then linearly on the remainder of the interval. Set I =(μ/4,1/2] and J =(λ/4,1/2], then since 0 is an attracting fixed point, the intervals L^n μ(I) and L^n λ(J) are disjoint for n ∈ Z+, and their union is all of (0,1/2]. Extend the definition of h so that it is defined on (0,1/2] by;

$h(x)=L^n\lambda \circ h \circ L^{-n}\mu\ (x)$, for $x \in L^n\mu(I)$.

We can now check that h is continuous and increasing on [0,1/2] when we set h(0) = 0. Now define h on (1/2,1] by setting h(1 − x)=1−h(x) for x ∈ [0,1/2), clearly giving a homeomorphism on [0,1]. Then

Lλ(h(1 − x)) = Lλ(1 − h(x)) = Lλ(h(x)) = h(Lμ(x)) = h(Lμ(1 − x)),

so that h is the required conjugation.

CONCLUSION

In this paper we have established linear topological conjugacy in between logistic map with tent map ,logistic type map [11]and a quadratic map along with their region of convergency. Also the transitivity of logistic map for μ=4 and we have discussed the fundamental domain of logistic map. From the above discussion we can conclude that the quadratic maps are linearly equivalent conjugacy with logistic map.

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