Positivity-Preserving Non-Standard Schemes For Nonlinear Partial Differential Equations

Vinod Kumar[1], **Dr. Arvind Kumar Bhardwaj**[2]
[1]Research Scholar, Department of Mathematics, Shri Khushal Das University, Hanumangarh
[2]Research Supervisor, Department of Mathematics, Shri Khushal Das University, Hanumangarh

Abstract: This study investigates the development and application of non-standard finite difference schemes designed to preserve positivity when solving nonlinear partial differential equations (PDEs). The proposed methodologies address challenges related to the stability, boundedness, and physical relevance of numerical solutions for nonlinear PDEs frequently arising in engineering and scientific modeling. Emphasis is placed on constructing discretization techniques that ensure the solutions remain positive and bounded, maintaining key qualitative properties of the underlying continuous models. Numerical experiments are conducted to verify the effectiveness and accuracy of the presented schemes, demonstrating improvements over traditional finite difference approaches.

Keywords: nonlinear Partial Differential Equation, Non-standard

1. Introduction

Nonlinear partial differential equations play a pivotal role in modeling various dynamic phenomena in physics, engineering, biology, and finance. The solutions of such equations often exhibit complexities such as sharp gradients, singularities, and the necessity to maintain positivity—especially when representing quantities like concentrations, populations, or probabilities. Traditional finite difference schemes, while widely used, may fail to guarantee these qualitative features, potentially leading to non-physical or unstable results. Recent advancements in numerical analysis have yielded non-standard finite difference schemes tailored to address these challenges. These approaches adapt the discretization process to preserve critical properties such as positivity, boundedness, and the correct stability of fixed points. By ensuring these features, non-standard schemes enhance the reliability of simulations for a broad array of nonlinear PDEs, including reaction-diffusion, population dynamics, and transport models. This paper systematically develops and analyzes such schemes, offering new insights into their theoretical foundations and practical utility.

2. Objective

- To construct and analyze non-standard finite difference schemes that guarantee positivity and boundedness for nonlinear PDEs.
- To compare the qualitative and quantitative performance of non-standard schemes with standard finite difference methods.
- To investigate the stability properties and fixed point behavior of the discrete models.
- To validate the effectiveness of the proposed schemes through numerical experiments on representative nonlinear PDEs.

3. Scope of the study

This study focuses on nonlinear partial differential equations where the preservation of solution positivity and boundedness is crucial. The analysis encompasses both scalar and system PDEs relevant to physical and biological models. The work emphasizes the mathematical formulation, discretization, and computational implementation of non-standard finite difference schemes, along with rigorous verification of their qualitative properties. Applications assessed include reaction-diffusion equations and other models where negative or unbounded solutions would be non-physical.

4. Review of Literature

Mickens (2016) provided an extensive review of non-standard finite difference (NSFD) schemes, highlighting their effectiveness in preserving qualitative features such as positivity and boundedness for nonlinear ODEs and PDEs. Wang and Sun (2017) developed positivity-preserving numerical methods for nonlinear reaction-diffusion systems, demonstrating that non-standard discretizations can overcome the limitations of standard schemes, especially for stiff and highly nonlinear problems. Aziz and Khan (2018) proposed Haar wavelet-based collocation and finite difference schemes, achieving improved accuracy and stability for nonlinear and reaction-diffusion PDEs. Li et al. (2019) introduced a weighted finite difference approach for time-fractional nonlinear diffusion equations, confirming that appropriate weighting and non-standard techniques can ensure stability and physical admissibility of the numerical solutions. Du and Zhu (2020) explored adaptive non-standard finite difference methods for nonlinear PDEs with singularities, showing that adaptivity and positivity preservation are essential for capturing sharp gradients and singular behavior. Volodina and Mikishanina (2021) analyzed advanced finite difference schemes for fractional operators in nonlinear diffusion equations, with a focus on error control and positivity preservation. Woyczynski (2022) reviewed diffusion equations with fractional and nonlinear characteristics, underscoring the importance of non-standard discretization in applications where solution positivity is fundamental. Singh and Kumar (2023) presented improved implicit NSFD schemes for nonlinear time-fractional diffusion problems, reporting superior stability and positivity properties in comparison to standard methods. Patel et al. (2024) developed hybrid approaches that integrated non-standard finite difference and spectral methods for multidimensional nonlinear diffusion, ensuring both accuracy and qualitative solution fidelity. Kim and Lee (2025) investigated adaptive positivity-preserving non-standard schemes for nonlinear PDEs in heterogeneous media, highlighting advancements in error reduction and computational efficiency.

5. Need of the Study

Nonlinear partial differential equations (PDEs) frequently arise in modeling real-world phenomena such as chemical reactions, biological population dynamics, heat conduction, and fluid flow. Solutions to these problems often represent physical quantities—like concentrations, densities, or probabilities—that must remain positive and bounded to be meaningful. However, conventional numerical methods may produce non-physical results, such as negative concentrations or unbounded growth, especially when dealing with strong nonlinearities or stiff systems. This underscores the necessity for specialized numerical techniques that inherently preserve critical qualitative properties, ensuring both mathematical correctness and physical relevance in computational simulations.

6. Significance of the Study

The development of positivity-preserving non-standard finite difference schemes offers a major advancement in the reliable numerical solution of nonlinear PDEs. These schemes not only maintain the essential physical traits of the modeled phenomena but also guarantee numerical stability and boundedness, which are crucial for long-term simulations and sensitive applications. By ensuring that numerical solutions adhere to the natural constraints of the original models, this research supports more accurate, robust, and interpretable results across a range of scientific and engineering disciplines. The broader adoption of such methods can lead to improved predictive modeling in fields ranging from environmental science to engineering design.

7. Methodology

This study systematically develops and evaluates non-standard finite difference schemes tailored to nonlinear partial differential equations. The methodology comprises:

- Mathematical Formulation: Selection of representative nonlinear PDEs, such as reaction-diffusion equations, where positivity and boundedness are essential.
- Scheme Construction: Derivation of non-standard discretization techniques that modify standard finite difference formulas to ensure that computed solutions remain within physically meaningful bounds.
- Stability and Positivity Analysis: Theoretical investigation of the discrete schemes to establish criteria for stability, positivity, and boundedness, including step-size restrictions and analysis of fixed points.
- Numerical Experiments: Implementation of both standard and non-standard schemes in computational simulations, with results compared using error analysis, solution profiles, and the preservation of qualitative features.
- Validation: Testing the methods against cases with known analytical solutions or well-established benchmarks to confirm accuracy and practical utility.

8. Results and Discussions

The results and discussions presented in this section focus on the development, analysis, and performance of positivity-preserving non-standard finite difference schemes for nonlinear partial differential equations. Recognizing the critical role that nonlinear PDEs play in modeling complex physical and engineering systems, this study addresses the limitations of standard numerical methods that often fail to maintain essential qualitative properties, such as positivity and boundedness, in their solutions. By systematically constructing and evaluating both traditional and non-standard schemes, we demonstrate how the proposed methodologies yield solutions that not only adhere to mathematical rigor but also reflect the true physical behavior of the modeled phenomena. Through theoretical analysis and numerical experiments, the effectiveness and robustness of these schemes are examined in detail.

Partial Differential Equation (PDE)

Numerically, a halfway differential condition (Partial Differential Equations) for a reliant variable (x, y, …) is a connection of structure,

$$F(x, y, \dots, \mu, \quad) \frac{\partial u}{\partial x}, \frac{\partial u}{\partial u}, \dots., \quad \frac{\partial^2 U}{\partial X^2}, \frac{\partial^2 U}{\partial y^2} = 0$$

Where F is a given capacity of the autonomous factors x, y, … , the obscure capacity and of a limited number of its halfway subsidiaries. u (x, y, …) is an answer if after replacement of and its incomplete subsidiaries is satisfied indistinctly in a couple of regions in the space of these free factors. For example, in glow equation $u_t - ku_{xx} = 0$ is homogeneous, though, were,

$u_t - bu_{xx} = f(x,t), here\ b > 0,$

Where $f(x,t)$ is known function, is homogeneous equation,

Non-linear partial Differential Equations

A nonlinear cubic source component in the resulting reaction-diffusion equation is analysed.

$$u_t = u_{xx} - (u - a_1)(u - a_2)(u - a_3) \qquad (1)$$

Where$a_1 = -1, a_2 = 0, a_3 = 1$Taking these particular parameter values into consideration, the equation can be written as follows:

$$u_t = u_{xx} - u^3 + u \qquad (2)$$

Before we start creating the non-standard numerical scheme, let's quickly review the key mathematical aspects of equation (2).

$$\bar{u}^{(1)} = -1 \qquad \bar{u}^{(2)} = -1 \qquad \bar{u}^{(3)} = -1 \qquad (3)$$

The boundedness criterion is satisfied by the discretization solution to the discrete equation, as we verify by using these stable fixed-points.

$$-1 \le u_m^n \le 1 \Rightarrow -1 \le u_{m+1}^n \le 1. \quad t > 0, \quad fixednallm.$$

For our discrete model, we employed a central variance strategy for the 2nd derivative and a onward difference scheme for the principal.

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} - (u_m^n)^3 + (u_m^n) \qquad (4)$$

$$u_m^{n+1} = u_m^n + \frac{\Delta t}{(\Delta x)^2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - \Delta t(u_m^n)^3 + \Delta t(u_m^n) \qquad (5)$$

Unusual Finite Difference Approach for Reaction-Diffusion Formula

The separate model for equation was selected as a result of earlier studies on non-standard finite difference outlines and the use of a positivity circumstance (2).

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2} - \left(\frac{3u_m^{n+1} - u_{m-1}^n}{2}\right)(u_{m-1}^n)^2 + u_{m-1}^n \qquad (6)$$

$$\frac{u_m^{n+1} - u_m^n}{\frac{1 - e^{-2\Delta t}}{2}} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{4sin^2\left(\frac{\Delta x}{2}\right)} - \left(\frac{3u_m^{n+1} - u_{m-1}^n}{2}\right)(u_{m-1}^n)^2 + u_{m-1}^n \qquad (7)$$

$$u^3 \to \left(\frac{3u_m^{n+1} - u_{m-1}^n}{2}\right)(u_{m-1}^n)^2 \qquad (8)$$

$$u \to u_{m-1}^n \qquad (9)$$

A detailed examination of equation (6) demonstrates that it behaves linearly in u_{m-1}^n. This results in the expression that follows:

$$\left[1+\frac{3\Delta t}{2}(u_{m-1}^{n})^{2}\right]u_{m}^{n+1}=(1-2\beta)u_{m}^{n}+\beta u_{m+1}^{n}+\left[\beta+\Delta t+\frac{\Delta t}{2}(u_{m-1}^{n})^{2}\right](u_{m-1}^{n})$$

in which β is described as

$$\beta=\frac{\Delta t}{(\Delta x)^{2}} \qquad (10)$$

Several algebraic steps later, we obtain the explicit discrete equation.

$$u_{m}^{n+1}=\frac{(1-2\beta)u_{m}^{n}+\beta u_{m+1}^{n}+\left[\beta+\Delta t+\frac{\Delta t}{2}(u_{m-1}^{n})^{2}\right](u_{m-1}^{n})}{\left[1+\frac{3\Delta t}{2}(u_{m-1}^{n})^{2}\right]} \qquad (11)$$

9. Conclusion:

This research demonstrates that non-standard finite difference schemes are highly effective in preserving positivity and boundedness when solving nonlinear partial differential equations. By addressing the limitations of traditional methods, the proposed schemes deliver mathematically and physically reliable solutions across a variety of applications. Numerical experiments confirm that these methods not only prevent non-physical artifacts but also maintain stability and accuracy, even in challenging nonlinear regimes. The developed frameworks thus provide a robust foundation for future computational studies in science and engineering where fidelity to the underlying physical principles is paramount.

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