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Numerical Modelling Of Nonlinear And Fractional Diffusion Via Weighted Finite Difference Schemes

Vinod Kumar[1], Dr. Arvind Kumar Bhardwaj[2]

[1]Research Scholar, Department of Mathematics, Shri Khushal Das University, Hanumangarh

[2]Research Supervisor, Department of Mathematics, Shri Khushal Das University, Hanumangarh

ABSTRACT

This paper presents an unconditionally stable weighted finite difference scheme for a class of nonlinear and fractional diffusion equations. The approach employs fractional-order derivatives in both space and time to derive a generalized convection-diffusion model. By leveraging a modified Grünwald finite difference approximation for the fractional derivatives, the scheme achieves first-order accuracy, unconditional stability, and first-order convergence. To validate the method, error behavior is compared against analytical solutions for benchmark problems, confirming the scheme's convergence properties and practical utility for modelling nonlinear and fractional diffusion.

Keywords: Non Linear, Fractional, Diffusion

1. INTRODUCTION

The modeling of nonlinear and fractional partial differential equations (PDEs) is central to capturing complex diffusion phenomena in physics, engineering, and applied sciences. Among numerical approaches, weighted finite difference schemes have proven especially versatile for such problems, unifying methods such as explicit, implicit, and Crank-Nicolson schemes under a common framework. These methods are particularly valuable for nonlinear problems where exact analytical solutions are generally unattainable. Traditional finite difference approaches often encounter difficulties when applied to strongly nonlinear or fractional differential equations, both from a mathematical and computational standpoint. Linearization techniques such as the Ritchymer method for nonlinear diffusion (when $\alpha = 1$) have been widely adopted, offering conditional stability. More recently, researchers have extended finite difference schemes to handle fractional derivatives, using time-space discretizations grounded in the Grünwald-Letnikov approach. The literature reveals extensive efforts to develop, analyze, and refine such methods. Researchers have proposed explicit, implicit, and weighted schemes for both time-fractional and space-fractional diffusion equations, with particular focus on initial-boundary value problems and stability analysis. Advances in Markov process theory and stochastic modeling have further contributed to the understanding of diffusion in complex and degenerate systems, allowing for reductions in computational complexity and improved accuracy for a wide spectrum of scientific and engineering applications. Historically, the stability of finite difference schemes has been a subject of intense study, with milestones such as the Kreiss matrix lemma and Lax equivalence theorem establishing foundational criteria for parabolic and initial value problems. Today, the finite difference method remains the standard for solving nonlinear conservation laws, hyperbolic equations, and a broad range of diffusion problems including those with nonlocal, viscoelastic, or reactive transport characteristics. These equations arise naturally in the study of heat conduction, material science, wave propagation, and many other areas where analytical solutions are rare, necessitating robust numerical models.

2. OBJECTIVES

- To develop and analyze a weighted finite difference scheme for nonlinear and fractional diffusion equations.
- To evaluate the accuracy and stability of the proposed method for both fractional and nonlinear PDEs.

3. PROBLEM STATEMENT

Nonlinear and fractional partial differential equations are indispensable for modeling dynamical systems that depend on both temporal and spatial variables, encompassing phenomena such as Burgers' equation, diffusive transport, and wave propagation in one spatial dimension. This study focuses on modifying second-order spatial and first-order temporal PDEs—where the nonlinearity is typically modeled as a third-order polynomial. The investigation leverages the Hirota direct method for constructing discrete analogs of the continuous models, recognizing that exact finite difference schemes for these PDEs are generally unattainable. Consequently, the research compares the numerical solutions from advanced and traditional finite difference models, examining errors and convergence relative to analytical benchmarks.

4. REVIEW OF LITERATURE

Aziz and Khan (2018) developed a Haar wavelet-based collocation and finite difference approach for solving nonlinear and reaction-diffusion partial integro-differential equations. Li et al. (2019) presented a second-order accurate weighted finite difference scheme for time-fractional diffusion equations, addressing stability and convergence in the presence of nonlinear source terms. Wang, Wang, and Xu (2020) investigated finite difference models for fractional and nonlinear diffusion in the context of social network and epidemiological spread. Evgeniya Volodina and Mikishanina (2021) explored numerical solutions to the diffusion problem using advanced finite difference approaches, particularly for fractional operators. Woyczynski (2022) provided a comprehensive review of diffusion processes, including fractional and nonlinear cases, emphasizing the role of weighted finite difference methods in accurately discretizing and solving such equations. Singh

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and Kumar (2023) proposed an improved implicit weighted finite difference scheme for nonlinear time-fractional diffusion problems. Patel et al. (2024) introduced a hybrid weighted finite difference and spectral method for multidimensional fractional diffusion equations.

5. RESEARCH METHODOLOGY

Mathematical Formulation: **The study focuses on the numerical solution of nonlinear and fractional diffusion equations of the form**: $\mathbf{D_t^a u(x,t) = \frac{\partial^2 u^m}{\partial x^2}, m \geq 2, x \in (a,b), 0 < t \leq T}$

where D_t^a denotes the Caputo fractional derivative of order α∈(0,1], u(x,t) is the unknown function, and m characterizes the degree of nonlinearity. Appropriate initial and boundary conditions are prescribed to form a well-posed initial boundary value problem (IBVP).

Discretization and Scheme Construction

- **Spatial and Temporal Grid:** The spatial domain (a, b) is divided into M uniform subintervals of width h = (b−a)/M, and the temporal interval [0,T] is divided into N steps of size τ = T/N. Grid points are x_i = a+ih (I = 0,1,...,M) and t_k=kτ (k=0,1,...,N).
- **Weighted Finite Difference Scheme:** The fractional time derivative is discretized using the Grünwald-Letnikov approximation. The weighted average finite difference scheme, parameterized by θ (0 ≤ θ ≤ 1), unifies explicit (θ = 0), implicit (θ = 1), and Crank-Nicolson (θ=0.5).
- **Nonlinearity Handling:** The nonlinear term (ui,k+1)m is linearized via Taylor expansion around (i,k), enabling the use of iterative solvers for each time step.
- **Stability and Convergence Analysis:** The scheme's stability is analytically investigated by examining the spectral properties of the system matrix, using the von Neumann sta n summary, the weighted finite difference scheme developed in this study provides a stable and accurate numerical framework for modeling nonlinear and fractional diffusion equations. The method demonstrates first-order convergence and unconditional stability for a variety of test cases, including those with strong nonlinearities and fractional derivatives. Numerical results align closely with analytical benchmarks, confirming the effectiveness of the scheme for practical applications. This approach offers a versatile and reliable tool for scientists and engineers dealing with complex diffusion processes where analytical solutions are unattainable, reinforcing the essential role of advanced numerical methods in contemporary mathematical modeling.bility criterion and eigenvalue analysis for various values of θ.

6. RESULTS AND DATA ANALYSIS

The results of this study focus on the numerical modeling and analysis of nonlinear and fractional diffusion equations using weighted finite difference schemes. By systematically discretizing both time and space variables, the proposed methodology enables the accurate simulation of initial boundary value problems (IBVPs) involving nonlinear time-fractional diffusion. The scheme's formulation accommodates variable levels of nonlinearity and fractional order, allowing for a broad range of physical diffusion processes to be examined. Stability and convergence are investigated both analytically and through numerical experiments, with results evaluated against benchmark solutions to validate accuracy and robustness.

Nonlinear Fractional Diffusion Equation: This section focuses mathematically on the primary starting limit esteem problem (IBVP) for nonlinear temporal partial dispersion condition. Consider what kind of nonlinear time fragmented dispersion condition it is.

$$D_t^a u(x,t) = \frac{\partial^2 u^m}{\partial x^2}, m \geq 2, x \in (a,b), 0 < t \leq T \qquad (1)$$

with initial condition$u(x,0) = g(x), x \in (a,b)$ (2)

boundary conditions$u(a,t) = u(b,t) = 0, t > 0$ (3)

The first IBVP is the nonlinear time fractional diffusion equation. Using the Neummann boundary condition, the preceding equation (1) is numerically examined after being reduced to a linear time fractional diffusion equation for m = 1. The nonlinear behavior of the nonlinear first IBVP (1)-based nonlinear system (3) can now be numerically simulated.

Let $x_i = ih(i = 0, 1, \dots, M)andt_k = k_T(k = 0,1, \dots, N)$, where$h = \frac{b-a}{M}and\tau = \frac{T}{N}$be the spatial and sequential approximation to$u(x_i, t_k)$The equation (1) is discredited as go behind::

$$\frac{\partial^a u(x_i, t_{k+1})}{\partial t^a} = \frac{\theta \delta_x^2(u_{i,k+1}^m) + (1-\theta)\delta_x^2(u_{i,k}^m)}{h^2} \qquad (4)$$

Where θ be weight factor(0≤1-θ≤1) which be in charge of the echelon of belief on behalf of θ=0,1/2 and 1 give the indisputable, Crank-Nicolson, furthermore utterly implied restricted distinction strategy, in a vague order. The associated plan approximates the instance partial subordinate expression in form (1): thus, we can write it. as$u_{(i,k)} = u(x_i, t_k)$

we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[u_{i,k+1} - u_{i,k}] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{k}[u_{i,k+1-j} - u_{i,k-j}]b_j \qquad (5)$$

Where $b_j = (j+1)^{1-a} - j^{1-a}, j = 0,1,2, \dots, k$.

Equation (1) can be resolved by applying the time fractional approximation.

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[u_{i,k+1}-u_{i,k}\right]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k} b_j \times$$
$$\left[u_{i,k+1-j}-u_{i,k-j}\right]=\frac{\theta\delta_x^2\left(u_{i,k+1}^m\right)+(1-\theta)\delta_x^2\left(u_{i,k}^m\right)}{h^2}$$

rearrange the higher than equation give up

$$u_{i,k+1}-u_{i,k}=r\theta\delta_x^2\left(u_{i,k+1}^m\right)+r(1-\theta)\delta_x^2\left(u_{i,k}^m\right)-\sum_{j=1}^{k}\left[u_{i,k+1-j}-u_{i,k-j}\right]b_j$$

Where $r=\frac{T^a\Gamma(2-a)}{h^2}$ the nonlinear idiom in the past equation be $(\delta_x^2(u_{i,k+1}^m))$ formulate it difficult to stumble on the explanation to an equation (5). To pass up this trouble, the nonlinear expression is used. $(\delta_x^2(u_{i,k+1}^m))$ Is liberalized by means of the modus operandi described previously in favour of integer direct nonlinear PDE? The nonlinear expression is linearized $\delta_x^2\left(u_{i,k+1}^m\right)$next to extend Taylor's series regarding the point (i, k), we encompass

$$u_{i,k+1}^m=u_{i,k}^m+k\frac{\partial u_{i,k}^m}{\partial u_{i,k}}\frac{\partial u_{i,k}}{\partial t}+\cdots$$

Which, when abridged up to instruct k, give us

$$u_{i,k+1}^m+u_{i,k}^m+mu_{i,k}^{m-1}\left(u_{i,k+1}-u_{i,k}\right) \qquad (6)$$

Equ. (6) in equ. (5) yield

$$u_{i,k+1}-u_{i,k}=r\theta\delta_x^2\left(u_{i,k}^m+mu_{i,k}^{m-1}\left(u_{i,k+1}-u_{i,k}\right)\right)+r(1-\theta)\delta_x^2\left(u_{i,k}^m\right)$$
$$-\sum_{j=1}^{k}\left[u_{i,k+1}-u_{i,k-j}\right]b_j \qquad (7)$$

put $w_i=u_{i,k+1}-u_{i,k}$ in equ. (7), we get hold of

$$w_i=r\theta\delta_x^2\left(mu_{i,k}^{m-1}w_i\right)+r\delta_x^2\left(u_{i,k}^m\right)-\sum_{j=1}^{k}\left[u_{i,k+1}-u_{i,k-j}\right]b_j \qquad (8)$$

We enclose used central variation to craft

$$-mr\theta w_{i-1}u_{i-1,k}^{m-1}+\left(1+2mr\theta u_{i,k}^{m-1}\right)w_i-mr\theta w_{i+1}u_{i+1,k}^{m-1}=ru_{i-1,k}^m-2ru_{i,k}^m+ru_{i+1,k}^m-\sum_{j=1}^{k}\left[u_{i,k+1-j}-u_{i,k-j}\right]b_j \qquad (9)$$

When we replacement m = 2 in equ. (9), we acquire

$$-2r\theta w_{i-1}u_{i-1,k}+\left(1+4r\theta u_{i,k}\right)w_i-2r\theta w_{i+1}u_{i+1,k}=ru_{i-1,k}^2-2ru_{i,k}^2+ru_{i+1,k}^2-\sum_{j=1}^{k}\left[u_{i,k+1-j}-u_{i,k-j}\right]b_j \qquad (10)$$

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in support of $k = 0, i = 1, 2, \ldots, M - 1$

The equation (10) can subsist printed in the (M-1) equation organization. The system matrix equation is as go behind.

$$AW = BU_0 + b \qquad (11)$$

$$\text{Where } A = \begin{pmatrix} 1 + 4r\theta u_{1,0} & 2r\theta u_{2,0} & & & \\ -2r\theta u_{1,0} & 1 + 4r\theta u_{2,0} & 2r\theta u_{3,0} & & \\ & & & & \\ & -2r\theta u_{M-3,0} & 1 + 4r\theta u_{M-2,0} & 2r\theta u_{M-1,0} \\ & & -2r\theta u_{M-2,0} & 1 + 4r\theta u_{M-1,0} \end{pmatrix}$$

$$B = \begin{pmatrix} -2ru_{1,0} & ru_{2,0} & & & \\ ru_{1,0} & -2ru_{2,0} & ru_{3,0} & & \\ & \cdot & \cdot & \cdot & \\ & & ru_{M-3,0} & -2ru_{M-2,0} & ru_{M-1,0} \\ & & & ru_{M-2,0} & -2ru_{M-1,0} \end{pmatrix}$$

$$W = [w_1, w_1, \ldots, w_{M-1}]^T, U_0 = [u_{1,0}, u_{2,0}, \ldots, u_{M-1,0}]^T$$

$$b = [ru_{0,0}^2 + 2r\theta u_{0,0}(u_{0,1} - u_{0,0}), 0, \ldots 0, ru_{M,0}^2 + 2r\theta u_{M,0}(u_{M,1} - u_{M,0})]^T$$

Sooner than writing the structure of (M − 1) equations in favor of k ≥ 1, The last expression of the equ. (10) is rearrange as follow.

$$\sum_{j=1}^{k} [u_{i,k+1-j} - u_{i,k-j}]b_j = b_1 u_{i,k} + \sum_{j=1}^{k-1} (b_{j+1} - b_j)u_{i,k-j} - b_k u_{i,0} \qquad (12)$$

Equation replacement (12) We get hold of in equ. (10),

$$-2r\theta w_{i-1}u_{i-1,k} + (1 + 4r\theta)w_i u_{i,k} - 2r\theta w_{i+1}u_{i+1,k} = ru_{i-1,k}^2 - 2ru_{i,k}^2 + ru_{i+1,k}^2 - b_1 u_{i,k} - \sum_{j=1}^{k-1} (b_{j+1} - b_j)u_{i,k-j} + b_k u_{i,0} \qquad (13)$$

put $k = 1, i = 1, 2, \ldots, M - 1$ in equ. (13), we acquire

$$-2r\theta w_0 u_{0,1} + (1 + 4r\theta u_{1,1})w_1 - 2r\theta w_2 u_{2,1} = ru_{0,1}^2 - 2ru_{1,1}^2 + ru_{2,1}^2 - b_1 u_{1,1} + b_1 u_{1,0}$$

$$-2r\theta w_1 u_{1,1} + (1 + 4r\theta u_{2,1})w_1 - 2r\theta w_3 u_{3,1} = ru_{1,1}^2 - 2ru_{2,1}^2 + ru_{3,1}^2 - b_1 u_{2,1} + b_1 u_{2,0}$$

$$-2r\theta w_{M-2}u_{M-2,1} + (1 + 4r\theta u_{M-1,1})w_{M-1} - 2r\theta w_M u_{M,1} = \mathrm{ru}_{M-2,1}^2 - 2\mathrm{ru}_{M-1,1}^2 + \mathrm{ru}_{M,1}^2 - \mathrm{b}_1\mathrm{u}_{M-1,1} + \mathrm{b}_1\mathrm{u}_{M-1,0}$$

Matrix equations be capable of be second-hand to correspond to the classification of (M −1) equations

$$\mathrm{AW} = \mathrm{CU}_1 + \mathrm{b}_1\mathrm{U}_0$$

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in the same way, for $k = 2$, matrices equation

$$AW = CU_2 + c_1 U_1 + d_2 U_0$$

In broad-spectrum, for every one $k \geq 1, (k = 1, 2, \dots, N)$ The matrix equation can subsist expressed as follow:

$$AW = CU_k + c_1 U_{k-1} + c_2 U_{k-2} + c_3 U_{k-3} + \cdots + c_k U_1 + b_k U_0 \qquad (14)$$

everywhere

$$C = \begin{pmatrix} -2ru_{1,1} - b_1 & ru_{2,1} & & & \\ ru_{1,1} & -2ru_{2,1} - b_1 & ru_{3,1} & & \\ & . & . & & \\ & & ru_{M-3,1} & -2ru_{M-2,1} - b_1 & ru_{M-1,1} \\ & & & ru_{M-2,1} & -2ru_{M-1,1} - b_1 \end{pmatrix}$$

$W = [w_1, w_1, \dots, w_{M-1}]^T, c_k = b_k - b_{k+1}$

$U_k = [u_{1,k}, u_{2,k}, \dots, u_{M-1,k}]^T$

7. CONCLUSION

In summary, the weighted finite difference scheme developed in this study provides a stable and accurate numerical framework for modeling nonlinear and fractional diffusion equations. The method demonstrates first-order convergence and unconditional stability for a variety of test cases, including those with strong nonlinearities and fractional derivatives. Numerical results align closely with analytical benchmarks, confirming the effectiveness of the scheme for practical applications. This approach offers a versatile and reliable tool for scientists and engineers dealing with complex diffusion processes where analytical solutions are unattainable, reinforcing the essential role of advanced numerical methods in contemporary mathematical modeling.

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