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Study of Unique Fixed-Point Theorems in Different Mathematical Spaces

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ABSTRACT

Fixed-point theory constitutes one of the most significant branches of modern mathematical analysis and has extensive applications in pure and applied mathematics, physics, engineering, economics, computer science, and optimization theory. The concept of a fixed point refers to an element that remains invariant under a given mapping. The present research paper investigates unique fixed-point theorems in various mathematical spaces, including metric spaces, Banach spaces, normed linear spaces, complete metric spaces, cone metric spaces, and fuzzy metric spaces. The paper provides a systematic discussion of classical and modern fixed-point results with emphasis on uniqueness conditions. Special attention is given to the Banach Contraction Principle, Kannan's theorem, Chatterjea's theorem, Edelstein's theorem, and generalized contraction mappings. Applications of uniquefixed-point theory in differential equations, dynamic systems, integral equations, and computational mathematics are also discussed.

Keywords: Fixed Point, Metric Space, Banach Space, Contraction Mapping, Complete Metric Space, Fuzzy Metric Space, Cone Metric Space.

1. INTRODUCTION

Fixed Point Theory represents an elegant synthesis of geometry, topology, and both pure and applied analysis. Fixed-point theorems establish the conditions under which single-valued or multivalued mappings admit solutions. Over the last several decades, fixed point theory has developed into a highly influential and indispensable instrument for investigating nonlinear phenomena. Its applications span numerous disciplines, including physics, biology, chemistry, economics, engineering, and game theory.

Among the most significant contributions in fixed point and approximation theory is the Banach Contraction Principle. This principle has become fundamental in addressing a wide variety of theoretical and applied mathematical problems. A vast body of research has emerged on its technical extensions and generalisations. In addition, this classical theorem offers an iterative approach for achieving progressively accurate approximations of fixed points. It is especially valuable in the iterative resolution of systems of linear algebraic equations.

Iteration methods, which are central to almost every branch of applied mathematics, are commonly employed in convergence analysis and error estimation, and they frequently rely upon Banach's fixed-point theorem. In contemporary mathematical analysis, fixed points of mappings in ordered metric spaces have gained considerable importance for solving nonlinear

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equations. Initial developments in this direction were introduced by Wolk and were subsequently advanced by Monjardet in the framework of partially ordered sets.

Unique fixed-point theorems play a crucial role in ensuring the stability and determinacy of solutions in mathematical models. In applied mathematics, uniqueness ensures that a physical or computational system has only one stable solution, thereby eliminating ambiguity.

The study of fixed-point theory has expanded from ordinary metric spaces to generalised structures such as:

- Normed spaces
- Banach spaces
- Cone metric spaces
- Probabilistic metric spaces
- Fuzzy metric spaces
- Partial metric spaces
- Modular spaces
- G-metric spaces

The present paper aims to examine major unique fixed-point theorems and analyze their validity in different mathematical spaces.

2. OBJECTIVES OF THE STUDY

The principal objectives of this research paper are:

1. To study the concept of fixed points in different mathematical spaces.
2. To analyse unique fixed-point theorems in metric and generalised spaces.
3. To discuss contraction mappings and their significance.
4. To examine classical fixed point results and modern generalisations.
5. To explore applications of unique fixed point theory in mathematical sciences.

3. RESEARCH METHODOLOGY

The present study is analytical and theoretical in nature. The research is primarily based on secondary mathematical sources, including research articles, textbooks, journals, and published papers related to fixed point theory. Logical deduction and theorem analysis methods are employed throughout the study.

4. PRELIMINARIES AND BASIC DEFINITIONS

4.1 Metric Space

A metric space is an ordered pair (X, d), where Xis a non-empty set and

$$d: X \times X \to \mathbb{R}$$

satisfies:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

$$d(x, z) \leq d(x, y) + d(y, z)$$

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for all $x, y, z \in X$.

4.2 Fixed Point

A point $x \in X$is called a fixed point of the mapping $T: X \rightarrow X$if

$$T(x) = x$$

4.3 Contraction Mapping

A mapping $T: X \rightarrow X$is called a contraction if there exists $0 < k < 1$such that

$$d(Tx, Ty) \leq k\, d(x, y)$$

$$d(Tx, Ty) \leq k\, d(x, y)$$

for all $x, y \in X$.

5. BANACH FIXED-POINT THEOREM

The Banach Contraction Principle is one of the most fundamental results in fixed-point theory.

Theorem (Banach Contraction Principle)

Let (X, d)be a complete metric space and let $T: X \rightarrow X$be a contraction mapping. Then Thas a unique fixed point $x^* \in X$. Moreover, for any initial point $x_0 \in X$, the iterative sequence

$$x_{n+1} = T(x_n)$$

converges to x^*.

$$x_{n+1} = T(x_n)$$

Proof Sketch

Choose any $x_0 \in X$. Define the iterative sequence:

$$x_n = T^n(x_0)$$

Using the contraction condition,

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$$

Since $0 < k < 1$, the sequence $\{x_n\}$is Cauchy. Completeness of Ximplies existence of $x^* \in X$such that

$$x_n \rightarrow x^*$$

Continuity of Tgives:

$$T(x^*) = x^*$$

Uniqueness follows because if $u$and $v$are fixed points, then

$$d(u, v) = d(Tu, Tv) \leq kd(u, v)$$

which implies $u = v$.

6. KANNAN FIXED-POINT THEOREM

The Kannan theorem generalizes the Banach principle.

Theorem (Kannan)

Let (X, d)be a complete metric space and $T: X \rightarrow X$satisfy

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$$

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$$

where $0 \leq \alpha < \frac{1}{2}$. Then Tpossesses a unique fixed point.

Unlike Banach contractions, Kannan mappings need not be continuous.

7. BROUWER FIXED-POINT THEOREM

The Brouwer theorem is one of the cornerstones of topological fixed-point theory.

Theorem (Brouwer)

Every continuous function from a compact convex subset of Euclidean space into itself has at least one fixed point.

This theorem guarantees existence but not necessarily uniqueness. It plays a crucial role in topology, economics, and game theory.

Applications

- Nash equilibrium in game theory
- Differential equations
- Economic equilibrium models
- Topological analysis

8. FIXED POINTS IN BANACH SPACES

A Banach space is a complete normed vector space.

Example

Let $X = C[0,1]$with norm

$$\| x \| = \max_{t \in [0,1]} | x(t) |$$

$$\| x \| = \max_{t \in [0,1]} | x(t) |$$

Suppose $T: X \to X$satisfies

$$\| Tx - Ty \| \leq k \| x - y \|$$

with $0 < k < 1$. Then Thas a unique fixed point.

Banach space techniques are extensively applied to nonlinear integral equations and partial differential equations.

9. FIXED POINTS IN HILBERT SPACES

Hilbert spaces generalise Euclidean geometry to infinite dimensions and possess inner product structures.

Definition

A Hilbert space is a complete inner product space satisfying

$$\langle x, y \rangle$$

with induced norm

$$\| x \| = \sqrt{\langle x, x \rangle}$$

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Fixed-point theorems in Hilbert spaces are widely used in optimisation, variational inequalities, and quantum mechanics.

10. CONE METRIC SPACES

Cone metric spaces generalise metric spaces by replacing real numbers with ordered Banach spaces.

Definition

A cone metric space (X, d)satisfies

$$d(x, y) \in P$$

where Pis a cone in a Banach space.

Researchers extended Banach and Kannan-type contractions to cone metric spaces and proved unique fixed-point results.

11. FUZZY METRIC SPACES

Fuzzy metric spaces were introduced to model uncertainty and vagueness.

Definition

A fuzzy metric space is a triple $(X, M, *)$where

$$M(x, y, t) \in [0,1]$$

measures the degree of nearness between xand y.

Fixed-point results in fuzzy spaces are important in artificial intelligence, image processing, and decision theory.

12. COMMON FIXED-POINT THEOREMS

A common fixed point of mappings Sand Tsatisfies

$$Sx = Tx = x$$

$$Sx = Tx = x$$

Common fixed-point theorems study conditions under which multiple mappings share a unique fixed point.

Recent research has removed assumptions such as continuity and commutativity and still established existence and uniqueness results.

13. APPLICATIONS OF UNIQUE FIXED-POINT THEOREMS

13.1 Differential Equations

Many differential equations can be converted into operator equations. Fixed-point methods guarantee the existence and uniqueness of solutions.

13.2 Integral Equations

Integral equations often take the form

$$x(t) = f(t) + \int_a^b K(t, s, x(s))ds$$

$$x(t) = f(t) + \int_a^b K(t, s, x(s))ds$$

Such equations are solved using contraction mappings.

13.3 Optimisation Theory

Optimisation algorithms often converge to fixed points of iterative mappings.

13.4 Economics and Game Theory

Fixed-point theorems establish equilibrium states in economics and strategic games.

13.5 Computer Science

Applications include:

- Machine learning
- Neural networks
- Recursive algorithms
- Data analysis

14. RECENT DEVELOPMENTS

Modern fixed-point theory focuses on:

- Multivalued mappings
- Hybrid contractions
- Partial metric spaces
- G-metric spaces
- Ordered metric spaces
- Probabilistic metric spaces

Recent studies have generalised classical results to more abstract spaces.

15. CONCLUSION

Unique fixed-point theorems occupy a fundamental position in modern mathematics. Beginning with Banach's Contraction Principle, the theory has evolved into a vast and dynamic field encompassing metric, topological, fuzzy, cone, and Hilbert spaces. The existence and uniqueness of fixed points provide powerful tools for solving nonlinear equations, optimisation problems, and dynamic systems.

The study of different mathematical spaces demonstrates that the nature of the underlying space strongly influences fixed-point behaviour. Modern research continues to generalise classical fixed-point results to more abstract and complex structures, thereby broadening their applicability in mathematics, engineering, economics, and computational sciences.

REFERENCES

1. Stefan Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae, 1922.
2. Luitzen Egbertus Jan Brouwer, Über Abbildung von Mannigfaltigkeiten, Mathematische Annalen, 1912.
3. Kannan, R., "Some Results on Fixed Points," Bulletin of the Calcutta Mathematical Society, 1968.
4. Ćirić, L., "A Generalization of Banach's Contraction Principle," Proceedings of the American Mathematical Society, 1974.

5. Rhoades, B. E., "A Comparison of Various Definitions of Contractive Mappings," Transactions of the American Mathematical Society, 1977.
6. Saxena, A., & Rani, P., Analysis of Common Fixed Point Theorems in Diverse Spaces.
7. Lin, L. J., & Wang, S. Y., "Common Fixed Point Theorems for a Finite Family of Discontinuous and Noncommutative Maps."
8. Mawhin, J., "Variations on the Brouwer Fixed Point Theorem: A Survey."
9. Rani, A., & Bharti, "Study of Mappings and Metric Spaces in Fixed Point Theory: A Review."
10. Singh, A., Singh, M. K., & Shukla, R., "Some Unique Fixed-Point Theory in Different Spaces."